HODOGRAPH METHODS APPLIED TO

FLOW PAST FINITE WEDGES

being a THESIS presented by

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to the University of St. Andrews in

application for the degree of

DOCTOR OF PHILOSOPHY.

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D E C L A R A T I O N.

I hereby declare that the following thesis is a record of original work, that it has been composed by me, and that it has not been accepted for any other degree.

PERSONAL FOREWORD.

This work was begun in October 1948 when I was admitted as a Research Student in University College, Dundee, to which I was appointed as a Lecturer in Mathematics. Previously I was at Edinburgh University where I graduated in July 1948 with First Class Honours in Mathematics and Natural Philosophy.

When I left Dundee in September 1950, the work was continued during vacations from the University of Cambridge and was completed in Manchester University in December 1952.

I am deeply indebted to Dr. D. C. Pack for introducing me to the subject and for his constant encouragement and advice. I also wish to thank Professor E. T. Copson for acting as my second supervisor.

The substance of Chapter III of this thesis was published as a joint paper by Dr. Pack and myself in the Proceedings of the Cambridge Philosophical Society (1).

D E C L A R A T I O N.

I certify that Mr. A. G. Mackie has fulfilled the conditions of the Ordinance and Regulations for the presentation of the following thesis.

Research Supervisor.

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CHAPTER I.

THE BASIC EQUATIONS.

The Fundamental Assumptions. The fluid in the following work is assumed to be continuous and permanent and to obey the perfect gas law.

If p, ρ and T are the pressure, density and temperature respectively of the gas then

$$p = R\rho T \tag{1.1}$$

where R is a constant. This is the equation of state for the gas.

Let dQ be the heat increment per unit mass added from an external source in a given process. Then if E is the internal energy of the gas per unit mass, the first law of Thermodynamics gives

$$dQ = dE + pd\left(\frac{1}{\rho}\right) \tag{1.2}$$

the last term denoting the work done in altering the shape of the element.

In general E is a function of p, ρ and T (or of any two of them through (1.1)). Experimental evidence shows that E is independent of the density when the temperature is kept constant.

Thus the differential relation

$$dE = \left(\frac{\partial E}{\partial \rho}\right)_{T=\text{const}} d\rho + \left(\frac{\partial E}{\partial T}\right)_{\rho=\text{const}} dT$$

may be written as $dE = C_v dT$ where $C_v = \left(\frac{\partial E}{\partial T}\right)_{\rho = \text{const}}$

or $\qquad E = C_v T \qquad (1.3)$

by suitable choice of origin of E.

(1.2) can now be rewritten as

$$dQ = C_v\, dT + p\, d\left(\frac{1}{\rho}\right). \qquad (1.4)$$

C_v is thus seen to be the specific heat at constant density, that is the ratio of the heat increment to the temperature increment when the density is kept constant. C_p is similarly defined for pressure and is given by

$$C_p = C_v + R. \qquad (1.5)$$

This is obvious when (1.4) is rewritten by means of (1.1) as

$$dQ = (C_v + R)\, dT - \frac{RT}{p}\, dp.$$

γ is defined as $\frac{C_p}{C_v}$, the ratio of specific heats. The value of γ may be taken as constant for temperatures below about 2000°K and is about 1.4 for air. This value has been chosen for the numerical work.

By means of (1.1) (1.3) and (1.5) it is easily seen that

$$E = \frac{1}{\gamma-1}\,\frac{p}{\rho}. \qquad (1.6)$$

Processes in which no heat is lost to or supplied by the boundaries of the element of fluid are called adiabatic. For such processes $dQ = 0$ and (1.2), (1.6) give

$$0 = d\left(\frac{1}{\gamma-1}\frac{p}{\rho}\right) + p\,d\left(\frac{1}{\rho}\right) \quad \text{or}$$

$$d\left(\frac{p}{\rho^\gamma}\right) = 0, \text{ that is}$$

$$\frac{p}{\rho^\gamma} = C, \text{ a constant.} \qquad (1.7)$$

The assumption of an adiabatic process implies that each particle of the gas moves isentropically. (For the entropy S is defined by the differential relation $\frac{dQ}{T} = dS$ which is a perfect differential as can be seen from (1.4)). The grounds for such an assumption lie in the supposition that the viscous forces leading to heat dissipation are negligible and that no heat is convected across the flow field. Experiments verify that this approximation is valid over the whole flow field outside the boundary layer with the exception of certain sharply defined regions, known as shock waves, where viscosity effects cause what is effectively a breakdown in the continuity of the velocity. Such regions are, however, severely isolated. The general principle in the following work will be to try and find a shock free solution of the problem. It will be seen that certain types of singularities in the solution suggest that the continuity of the flow breaks down and the occurrence of a shock wave might then be postulated. This problem will be discussed qualitatively when it arises in a later section.

It is further assumed that all the streamlines emanate from a region of uniform conditions. The assumption that each particle of the gas moves isentropically then implies constant entropy and hence the invariance of $\frac{p}{\rho^\gamma}$ over the whole field.

<u>Equation of Continuity.</u> Consider a fixed volume of space occupied by the fluid. Since the fluid is permanent, the rate of increase of the mass contained in this volume must be equal to the rate at which mass flows in over the boundary. That is

$$\frac{\partial}{\partial t}\left\{\int \rho \, dV\right\} = -\int \rho \, \underline{q} \cdot \underline{n} \, dS$$

where $\underline{q}$ is the velocity vector of the flow and $\underline{n}$ the outward unit vector normal to the surface bounding the fixed volume.

Since $\frac{\partial}{\partial t}\left\{\int dV\right\} = 0$, this becomes, because of the divergence theorem

$$\int \left\{\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underline{q}\right\} dV = 0.$$

Since this is true for an arbitrary volume the integrand must vanish. That is

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underline{q} = 0 \qquad (1.8)$$

$$\text{or} \quad \frac{D\rho}{Dt} + \rho \operatorname{div} \underline{q} = 0. \qquad (1.8')$$

This is the equation of continuity. $\frac{D}{Dt}$ denotes differentiation following the motion of the fluid and is equivalent to the operator $\frac{\partial}{\partial t} + \underline{q} \cdot \nabla$.

Equation of Motion. If there is no external force field acting and as viscosity effects are to be neglected, the motion of an element of the fluid will be determined by the pressure forces on the surface bounding the element according to the equation

$$\frac{D}{Dt}\left\{ \int \rho \underline{q}\, dV \right\} = - \int p \underline{n}\, dS$$

where the integration is now taken over a volume moving with the fluid. Since $\frac{D}{Dt}\left\{ \int \rho\, dV \right\} = 0$ by continuity this reduces to

$$\int \left\{ \rho \frac{D\underline{q}}{Dt} + \text{grad } p \right\} dV = 0.$$

Since the volume is again arbitrary the integrand must vanish and

$$\rho \frac{D\underline{q}}{Dt} + \text{grad } p = 0. \qquad (1.9)$$

Circulation and Vorticity. The circulation round a closed curve C in the fluid is defined by the line integral $\oint_C \underline{q} . d\underline{s}$ The rate of change of circulation is therefore $\frac{D}{Dt}\{\oint_C \underline{q} . d\underline{s}\}$.

Now $$\frac{D}{Dt}\left\{ \oint_C \underline{q} . d\underline{s} \right\} = \oint_C \frac{D\underline{q}}{Dt} . d\underline{s} + \oint_C \underline{q} . \frac{D(d\underline{s})}{Dt}$$

$$= - \oint_C \frac{\text{grad } p}{\rho} . d\underline{s} + \left[\tfrac{1}{2} q^2\right]_C$$

Because of the functional relationship between p and ρ which holds for the whole field (the adiabatic law), and the single-valuedness of the velocity, the right hand side vanishes. It follows immediately that

$$\text{curl } \underline{q} = 0 \tag{1.10}$$

the field being vortex free initially, that is in the flow at infinity upstream. Thus $\underline{q}$ can be written as the gradient of a single valued potential ϕ , that is

$$\underline{q} = \text{grad } \phi \tag{1.11}$$

If the motion is steady $\frac{\partial}{\partial t} = 0$ and (1.9) becomes

$$(\underline{q}.\nabla)\,\underline{q} + \frac{\text{grad } p}{\rho} = 0 \,. \tag{1.12}$$

That is $\nabla\left\{\frac{1}{2}q^2 + \int \frac{dp}{\rho}\right\} = 0$ since $(\underline{q}.\nabla)\,\underline{q} = \nabla\left(\frac{1}{2}\underline{q}^2\right) - \underline{q}\wedge \text{curl } \underline{q}$ and curl $\underline{q} = 0$.

Hence $\frac{1}{2}q^2 + \int \frac{dp}{\rho}$ is constant over the whole field.

This is the more general form of Bernouilli's equation in classical hydrodynamics and by means of (1.1) and (1.7) it may be written in a variety of forms. A common form is obtained by introducing a, the speed of sound in the gas, which is given by $\sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}$.

Thus $\frac{1}{2}q^2 + \frac{a^2}{\gamma - 1}$ = constant.

If the gas is supposed to come from a reservoir where it is at rest and where the velocity of sound is a_0 this becomes

$$\frac{1}{2}q^2 + \frac{a^2}{\gamma-1} = \frac{a_0^2}{\gamma-1} . \tag{1.13}$$

It is seen from this that the gas can be accelerated only up to a limiting velocity q_m say, corresponding, since $a \propto \sqrt{T}$, to the zero of absolute temperature and given by $q_m^2 = \frac{2a_0^2}{\gamma-1}$.

An alternative form is obtained by defining a* as the velocity of the gas when it is equal to the local velocity of sound. This gives

$$\frac{1}{2}q^2 + \frac{a^2}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)} a^{*2} . \tag{1.13'}$$

The Mach number M is defined by the ratio $\frac{q}{a}$. Clearly (1.13) or (1.13') can be used to give M in terms of q or a.

<u>Two-dimensional Motion</u>. Let u, v be the Cartesian components of the velocity in a two-dimensional motion, the polar coordinates being q and θ . If the motion is steady (1.8) becomes

$$\operatorname{div}(\rho \underline{q}) = 0 . \tag{1.14}$$

As this implies $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$, a function Ψ exists such that

$$u = \frac{\rho_0}{\rho}\frac{\partial \Psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

$$v = -\frac{\rho_0}{\rho}\frac{\partial \Psi}{\partial x} = \frac{\partial \phi}{\partial y} .$$

ρ_0 is some constant density, inserted for dimensional purposes, which is taken as the density of the gas when it is decelerated isentropically to zero velocity.

Since
$$d\Psi = \frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy$$
$$= \frac{\rho}{\rho_0}\left(-v\,dx + u\,dy\right)$$
$$= 0 \quad \text{on a streamline (where } \frac{dy}{dx} = \frac{v}{u}\text{)},$$
Ψ has a constant value on each streamline and is known as the stream function.

An equation for ϕ is obtained by eliminating p and ρ from (1.12), (1.14) and the relation $\frac{dp}{d\rho} = a^2$.

Now
$$\underline{q}\cdot(\underline{q}\cdot\nabla\underline{q}) = -\frac{\underline{q}\cdot\text{grad}\,p}{\rho} \quad \text{from (1.12)}$$
$$= -\frac{\underline{q}\cdot\text{grad}\,\rho\;a^2}{\rho}$$

and
$$\underline{q}\cdot\text{grad}\,\rho = -\rho\,\text{div}\,\underline{q} \quad \text{from (1.14).}$$

Thus
$$\underline{q}\cdot(\underline{q}\cdot\nabla\underline{q}) = a^2\,\text{div}\,\underline{q}.$$

That is
$$(u^2-a^2)\frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (v^2-a^2)\frac{\partial v}{\partial y} = 0$$

or, with $\underline{q} = \text{grad}\,\phi$,
$$(u^2-a^2)\frac{\partial^2\phi}{\partial x^2} + 2uv\frac{\partial^2\phi}{\partial x\partial y} + (v^2-a^2)\frac{\partial^2\phi}{\partial y^2} = 0 \qquad (1.15)$$
where a^2 is a function of $u^2 + v^2$ related to it by (1.13).

Discussion of the Equation. Equation (1.15) is fundamental in the two-dimensional theory of the steady motion of a perfect, inviscid gas. It is a second order partial differential equation of the first degree but non-linear since the coefficients of the second order derivatives involve u and v, the first order derivatives of the dependent variable ϕ. The type of equation is determined by the sign of the discriminant. Thus the equation is of hyperbolic or elliptic type according as $u^2v^2 >$ or $< (u^2-a^2)(v^2-a^2)$; that is, according as $M >$ or < 1.

In other words, where the flow is subsonic the equation is elliptic and where the flow is supersonic the equation is hyperbolic. The fundamental mathematical problem in dealing with transonic flows is that the differential equation for such flows is of mixed type and this presents a formidable analytical difficulty.

Laplace's equation and the wave equation (each considered with two independent variables) are the simplest and best known examples of the two types. They illustrate an important difference between the types of solution obtained. For elliptic equations any perturbation of a given solution results in a perturbation over the entire field. This is not necessarily so in a hyperbolic differential equation. For example an obstacle introduced into the uniform potential flow field of an incompressible fluid affects the velocity at every part of the field. But in a one-dimensional wave (e.g. along a string where x and t are the independent variables) a disturbance started at one end takes a finite time to reach the other end. Consequently a section of the x,t plane is completely unaffected by the disturbance.

This property is well illustrated in the supersonic flow of a gas and admits an easy physical interpretation. Suppose a uniform supersonic gas stream meets a small obstacle. Since the obstacle signifies its presence by means of pressure pulses which travel relative to the gas at the local speed of sound, it follows that the gas in front of the obstacle is not affected by its presence; the disturbance is confined to the downstream side of the obstacle. If a small disturbance is introduced at a point in a uniform supersonic gas stream, simple geometry shows that the effects of the disturbance are confined within a cone of apex angle $2 \sin^{-1} \frac{1}{M}$ whose apex is at the point and whose axis is parallel to the direction of the uniform stream. When conditions in a meridian section are considered, the arms of the angle are known as the "Mach lines" at the point.

A property of hyperbolic equations is that their characteristics are real and that a given solution may be altered quite legitimately by "patching" a suitable solution to the original, provided that this is done along a character-

-istic curve. A simple example is that of supersonic flow round a convex bend. This is obtained from the uniform flow solution by adding suitable new solutions along the characteristics. The limiting case of this is the famous Prandtl-Meyer expansion for flow round a sharp corner. It is necessary to ensure that the velocity remains continuous across the characteristic but the velocity derivatives need not. The significance of the possibilities opened by this approach in the present problem will be apparent later.

Equation (1.15) admits of few exact solutions of any physical interest due to the intrinsic difficulty of solution in a general form. Approximate methods yield a number of solutions which are in good agreement with observation in certain cases. Liepmann and Puckett (2) give some simple examples of "linearized flow" in which the motion is regarded as a first order perturbation of a unifirm stream. But these approximations are invalid in the neighbourhood of $M = 1$.

The Hodograph Transformation. A considerable advance was made about the end of the last century by Molenbroek (3) and Chaplygin (4) by transforming the equation so that the velocity coordinates q and θ replaced x and y as independent variables. The resulting equation is linear and has an exact solution in terms of hypergeometric functions.

The transformation is carried out as follows. It is convenient to use Ψ , the stream function as the dependent variable in the new equation.

$$d\phi = q\cos\theta\,dx + q\sin\theta\,dy = \frac{\partial\phi}{\partial q}dq + \frac{\partial\phi}{\partial\theta}d\theta$$

$$\frac{\rho_0}{\rho}d\Psi = -q\sin\theta\,dx + q\cos\theta\,dy = \frac{\rho_0}{\rho}\frac{\partial\Psi}{\partial q}dq + \frac{\rho_0}{\rho}\frac{\partial\Psi}{\partial\theta}d\theta$$

$$\therefore dx = \left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial q} - \frac{\rho_0}{\rho q}\sin\theta\frac{\partial\Psi}{\partial q}\right)dq + \left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial\theta} - \frac{\rho_0}{\rho q}\sin\theta\frac{\partial\Psi}{\partial\theta}\right)d\theta \qquad (1.16)$$

$$dy = \left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial q} + \frac{\rho_0}{\rho q}\cos\theta\frac{\partial\Psi}{\partial q}\right)dq + \left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial\theta} + \frac{\rho_0}{\rho q}\cos\theta\frac{\partial\Psi}{\partial\theta}\right)d\theta\,. \qquad (1.17)$$

These give $\frac{\partial x}{\partial q}$, $\frac{\partial x}{\partial\theta}$, $\frac{\partial y}{\partial q}$ and $\frac{\partial y}{\partial\theta}$ and the equivalence of $\frac{\partial}{\partial q}\left(\frac{\partial x}{\partial\theta}\right)$ and $\frac{\partial}{\partial\theta}\left(\frac{\partial x}{\partial q}\right)$ requires

$$-\frac{\sin\theta}{q}\frac{\partial\phi}{\partial q} - \frac{\rho_0}{\rho q}\cos\theta\frac{\partial\Psi}{\partial q} = -\frac{\cos\theta}{q^2}\frac{\partial\phi}{\partial\theta} - \sin\theta\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial q}\left(\frac{\rho_0}{\rho q}\right).$$

Similarly $$\frac{\cos\theta}{q}\frac{\partial\phi}{\partial q} - \frac{\rho_0}{\rho q}\sin\theta\frac{\partial\Psi}{\partial q} = -\frac{\sin\theta}{q^2}\frac{\partial\phi}{\partial\theta} + \cos\theta\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial q}\left(\frac{\rho_0}{\rho q}\right).$$

These simplify to

$$\frac{1}{q}\frac{\partial\phi}{\partial q} = \frac{\partial}{\partial q}\left(\frac{\rho_0}{\rho q}\right)\frac{\partial\Psi}{\partial\theta} \qquad (1.18)$$

$$\frac{1}{q^2}\frac{\partial\phi}{\partial\theta} = \frac{\rho_0}{\rho q}\frac{\partial\Psi}{\partial q}\,. \qquad (1.19)$$

Now an equivalent form of equation (1.13) is

$$\frac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p_0\rho^{\gamma-1}}{\rho_0^{\gamma}} = \frac{a_0^2}{\gamma-1}$$

$$\therefore \quad q\,dq + \frac{\gamma p_0\rho^{\gamma-2}}{\rho_0^{\gamma}}d\rho = 0$$

$$\therefore \quad \frac{dq}{d\rho} = -\frac{1}{q}\,\frac{\gamma p_0 \rho^{\gamma-2}}{\rho_0^{\gamma}}$$ (the fact that ρ is independent of θ has already been used in the analysis).

$$\therefore \quad \frac{\partial}{\partial q}\left(\frac{\rho_0}{\rho q}\right) = -\frac{\rho_0}{\rho q^2} + \frac{\rho_0}{q\rho^2}\,\frac{q\rho_0^{\gamma}}{\gamma p_0 \rho^{\gamma-2}}$$

or $$\frac{\partial}{\partial q}\left(\frac{\rho_0}{\rho q}\right) = -\frac{\rho_0}{\rho}\left(\frac{1}{q^2} - \frac{1}{a^2}\right) \tag{1.20}$$

Thus $$\frac{\partial \phi}{\partial q} = \frac{\rho_0}{\rho q}\left(M^2 - 1\right)\frac{\partial \Psi}{\partial \theta} \tag{1.21}$$

$$\frac{\partial \phi}{\partial \theta} = \frac{q\rho_0}{\rho}\,\frac{\partial \Psi}{\partial q}. \tag{1.22}$$

From these equations ϕ has to be eliminated to give $\frac{1}{q}\,\frac{\rho_0}{\rho}\,(M^2-1)\,\frac{\partial^2 \Psi}{\partial \theta^2} = \frac{q\rho_0}{\rho}\left[\frac{\partial^2 \Psi}{\partial q^2} + \frac{\partial \Psi}{\partial q}\left\{q\left(\frac{1}{a^2} - \frac{1}{q^2}\right) + \frac{2}{q}\right\}\right]$

or $$q^2\,\frac{\partial^2 \Psi}{\partial q^2} + q\left(1 + \frac{q^2}{a^2}\right)\frac{\partial \Psi}{\partial q} + \left(1 - \frac{q^2}{a^2}\right)\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

q^2 and a^2 are related by the equation $q^2 + 2\beta a^2 = q_m^2$ where $\beta = \frac{1}{\gamma - 1}$.

If now the substitution $\tau = \frac{q^2}{q_m^2}$ is made* the resulting equation is

$$4\tau^2(1-\tau)\frac{\partial^2 \Psi}{\partial \tau^2} + 4\tau\{1 + (\beta-1)\tau\}\frac{\partial \Psi}{\partial \tau} + \{1 - (2\beta+1)\tau\}\frac{\partial^2 \Psi}{\partial \theta^2} = 0. \tag{1.23}$$

If a solution of the form $T(\tau)\,\Theta(\theta)$ is sought, the separate equations for T and Θ are

$$\Theta'' + n^2\,\Theta = 0$$ and

$$4\tau^2(1-\tau)T'' + 4\tau\{1 + (\beta-1)\tau\}T' - n^2\{1 - (2\beta+1)\tau\}T = 0. \tag{1.24}$$

* See foot of next page.

The substitution $G(\tau) = \tau^{-\frac{n}{2}} T(\tau)$ is now made and the resulting equation for $G(\tau)$ is

$$\tau(1-\tau)\, G''(\tau) + \{(n+1) - (n+1-\beta)\tau\}\, G'(\tau) + \frac{n(n+1)\beta}{2} G(\tau) = 0. \quad (1.25)$$

This will have two linearly independent solutions of which one is readily identified as the hypergeometric function $F(a_n, b_n; n+1; \tau)$ where $a_n + b_n = n - \beta$ and $a_n b_n = -\frac{\beta n(n+1)}{2}$. The expression $\tau^{\frac{n}{2}} F(a_n, b_n; n+1; \tau)$ will be denoted by $\psi_n(\tau)$. The other solution will in general be singular at $\tau = 0$ and a complete solution of (1.25) is given by a linear combination of the two.

Since, however, (1.24) is obviously satisfied by $\psi_{-n}(\tau)$ (and this will in general be an independent solution since near $\tau = 0$ $\psi_{\pm n}(\tau) \sim \tau^{\pm\frac{n}{2}}$) the complete solution of (1.23) may be written as

$$\Psi = \sum_n A_n \psi_n(\tau) \sin(n\theta + \alpha_n) \quad (1.26)$$

where A_n, α_n are arbitrary constants.

In fact, $\psi_n(\tau)$ does not exist for integral values

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* Although q is the actual velocity, the "velocity τ" will sometimes be referred to where no confusion arises.

of $n < -1$ as the coefficients become infinite. However this may be accepted for the present as a formal solution. Apart from this, n can be completely arbitrary but it will be real if solutions are to be obtained in terms of harmonic functions. It is not necessary however to take n integral since a solution in the whole flow plane will correspond to a section of the plane of the velocity variables. This plane is commonly referred to as the hodograph plane.

Position Coordinates. Such a solution has little inherent value without reference to the physical plane. It is necessary to find x and y as functions of τ and θ . When this is done lines of constant τ and constant θ can at once be obtained since, along them, x and y are functions of a single parameter; the streamlines can also be found by relating the values of τ and θ which satisfy the relation $\Psi = \Psi(\tau,\theta) = \Psi_0$.

The position coordinates corresponding to the elementary solution $\psi_n(\tau)\sin(n\theta+\alpha)$ will now be obtained explicitly. It is assumed that $n \neq \pm 1$.

The potential function ϕ corresponding to $\Psi = \psi_n(\tau)\sin(n\theta+\alpha)$ is first found. This is determined from (1.21) and (1.22) as $\phi = -2\tau\frac{\rho_0}{\rho_n}\psi_n'(\tau)\cos(n\theta+\alpha)$ apart from an arbitrary constant.

From (1.16) and (1.17) with $z = x+iy$

$$dz = \left(\frac{e^{i\theta}}{q}\frac{\partial\phi}{\partial q} + \frac{i\rho_0}{\rho q}e^{i\theta}\frac{\partial\Psi}{\partial q}\right)dq + \left(\frac{e^{i\theta}}{q}\frac{\partial\phi}{\partial\theta} + \frac{i\rho_0}{\rho q}e^{i\theta}\frac{\partial\Psi}{\partial\theta}\right)d\theta \quad (1.27)$$

$$\therefore \frac{\partial z}{\partial\theta} = \frac{\rho_0}{\tau^{\frac{1}{2}} q_m \rho} e^{i\theta}\left\{2\tau\,\psi_n'(\tau)\sin(n\theta+\alpha) + i n\,\psi_n(\tau)\cos(n\theta+\alpha)\right\}$$

$$= \frac{i\rho_0}{\tau^{\frac{1}{2}} q_m \rho}\left\{-\tau\psi_n'(\tau)\left(e^{i(n\theta+\theta+\alpha)} - e^{i(-n\theta+\theta-\alpha)}\right) + \frac{n}{2}\psi_n(\tau)\left(e^{i(n\theta+\theta+\alpha)} + e^{i(-n\theta+\theta-\alpha)}\right)\right\}.$$

$$\therefore z = \frac{\rho_0}{\tau^{\frac{1}{2}} q_m \rho}\left\{\frac{e^{i(n\theta+\theta+\alpha)}}{n+1}\left(\frac{n\psi_n(\tau)}{2} - \tau\psi_n'(\tau)\right) - \frac{e^{i(-n\theta+\theta-\alpha)}}{n-1}\left(\frac{n\psi_n(\tau)}{2} + \tau\psi_n'(\tau)\right)\right\} + g(\tau).$$

Now $\quad q^2 + 2\beta a^2 = 2\beta a_0^2 = q_m^2.$

$$\therefore \tau + \frac{a^2}{a_0^2} = 1, \quad i.e.\ \frac{\rho}{\rho_0} = (1-\tau)^{\beta}.$$

$$\therefore z = \frac{1}{q_m\tau^{\frac{1}{2}}(1-\tau)^{\beta}}\left\{\frac{e^{i(n\theta+\theta+\alpha)}}{n+1}\left(\frac{n\psi_n(\tau)}{2} - \tau\psi_n'(\tau)\right) - \frac{e^{i(-n\theta+\theta-\alpha)}}{n-1}\left(\frac{n\psi_n(\tau)}{2} + \tau\psi_n'(\tau)\right)\right\} + g(\tau).$$

Substitution of this value of z in the expression for $\frac{\partial z}{\partial\tau}$ obtained from (1.27) gives, after some algebra, $g'(\tau) = 0$. Hence apart from an arbitrary constant

$$z = \frac{1}{q_m\tau^{\frac{1}{2}}(1-\tau)^{\beta}}\left\{\frac{e^{i(n\theta+\theta+\alpha)}}{n+1}\left(\frac{n}{2}\psi_n(\tau) - \tau\psi_n'(\tau)\right) - \frac{e^{i(-n\theta+\theta-\alpha)}}{n-1}\left(\frac{n}{2}\psi_n(\tau) + \tau\psi_n'(\tau)\right)\right\} \quad (1.28)$$

The position coordinates when Ψ is obtained by the superposition of such elementary solutions are found by the linear combination of expressions such as (1.28).

<u>Limitations of the Method.</u> It might be thought that the hodograph method provides a complete answer to the problem caused by the non-linearity of the compressible flow equation in the x, y coordinates. However, one of the major difficulties which arises is that a given problem in the physical plane provides little clue as to which particular combination of solutions in the hodograph variables should be chosen. In other words, a given solution of the linear differential equation in the velocity coordinates does give in general a permissible flow in the x, y plane but the boundary conditions in this plane are then determined from the solution <u>a posteriori</u>. There is therefore no general technique for solving a given problem by this method.

Moreover, certain solutions disappear completely in the course of the transformation. The simplest case is that of uniform flow. The whole of the physical plane is represented by a single point in the hodograph plane and to speak of a solution in the hodograph plane is therefore meaningless. Another example is that of supersonic flow, originally uniform, which is rounding a convex bend. The velocity variables are always related by an equation $q = q(\theta)$; the whole flow is therefore represented by a single curve in the hodograph plane.

The reason for these exceptions is that it has been tacitly assumed up to now that given regions in the hodograph and physical planes could be mapped non-singularly into one another. In using the hodograph transformation very great attention must be paid to the singularities of the transformation; that is to the points, lines or regions in which the Jacobian $J = \frac{\partial(x,y)}{\partial(u,v)}$ becomes zero or infinite. There is now a considerable literature discussing such singularities and an exhaustive account of the various possible types has been given by Craggs (5). The most important singularity which occurs in the present work is the so-called limit line where streamlines in the physical plane are found to turn back on themselves in a physically unacceptable manner. This has been interpreted as an analytical warning that shock free flow is no longer possible, although care must be taken not to identify the position of the limit line with that of the shock wave. The whole question of singularities will be discussed more fully in Chapter IV.

C H A P T E R   II.

APPLICATION OF THE HODOGRAPH TRANSFORMATION.

Some Exact Solutions. The first application of the hodograph method was made by Chaplygin (4) who used it to solve the problem of a gas jet issuing from an orifice in a plane wall. He also produced a solution which represented a jet hitting a plate of finite width and breaking off the edge with a free streamline. However, it was not until comparatively recently that any extensive use has been made of the method.

The solutions of (1.25) are in general infinite power series in τ and it is therefore natural to see if there are any exceptions to this. The only two possibilities are

(i) that the series $\psi_n(\tau)$ and $\psi_{-n}(\tau)$ can be combined so as to give a finite number of terms. This requires that, after a certain number of terms, the series shall be identical.

(ii) That the series for $\psi_n(\tau)$ itself terminates.

Cases $n = 0, \pm 1$. (i) It has already been mentioned that for $n = -2, -3, \ldots$ the terms in the series for $\psi_n(\tau)$

become infinite. The second solution corresponding to integral values of $n > 1$ has a logarithmic singularity at the origin and consequently there is no possibility of combining it in the required manner with $\psi_n(\gamma)$. Moreover for other values of n it is necessary that the powers $\gamma^{\frac{n}{2}+r}$ and $\gamma^{-\frac{n}{2}+s}$ should be equal for some positive integral values of r and s. This requires n to be integral. The only possibilities are therefore $n = 0$ and $n = 1$.

(a) $n = 0$. $\psi_0(\gamma)$ is immediately identified as 1. The second solution is obtained by means of (1.25) with $n = 0$. This gives

$$\gamma(1-\gamma)\,G''(\gamma) + \{1-(1-\beta)\gamma\}\,G'(\gamma) = 0$$

and consequently

$$G(\gamma) = A\int^{\gamma}\frac{d\gamma}{\gamma(1-\gamma)^{\beta}} + B$$

which is effectively a $\log\gamma$ term followed by an infinite series.

The corresponding value of Θ is $\Theta = P\theta + Q$.

Thus the only really simple solution for $n = 0$ is $\Psi = P\theta$ and this represents a form of spiral flow first examined by Taylor (6).

(b) $n = 1$. By writing out the series for $\psi_1(\gamma)$ and $\psi_{-1}(\gamma)$ it is easily seen that $\psi_{-1}(\gamma) - \frac{\beta}{2}\psi_1(\gamma) = \gamma^{-\frac{1}{2}}$.

This solution may be recovered more easily by putting $n = -1$ in (1.25).

$$\text{Then} \quad \tau(1-\tau)\, G''(\tau) + \beta\tau\, G'(\tau) = 0$$

$$G(\tau) = A + B(1-\tau)^{\beta+1}$$

With $n^2 = 1$, $\Theta = \sin(\theta + \varepsilon)$.

There are therefore two solutions,

$$\Psi = \tau^{-\frac{1}{2}} \sin\theta \quad \text{and} \quad \Psi = \tau^{-\frac{1}{2}} (1-\tau)^{\beta+1} \sin\theta .$$

The simplicity of the second solution is due to the hypergeometric series degenerating into an ordinary binomial expansion.

<u>Ringleb's Solution.</u> The solution $\Psi = \tau^{-\frac{1}{2}} \sin\theta$ was first discovered in Germany by Ringleb in 1940 (7). Due, however, to the inaccessibility of his paper at the time in this country, Temple and Yarwood published a translation of the substance of his work in 1942 (8) and also considered the second solution. The physical significance of the solution was discussed by von Kármán (9).

Ringleb's solution will be examined briefly as it presents features similar to those which occur in the more complicated solutions discussed later but which can be more easily identified here because of the relative simplicity of the algebra.

The value of ϕ corresponding to $\Psi = \tau^{-\frac{1}{2}} \sin\theta$ is $\phi = \tau^{-\frac{1}{2}} (1-\tau)^{-\beta} \cos\theta$. The position coordinates are found in an entirely analogous way to that in which they were found for $\psi_n(\tau)\sin(n\theta+\alpha)$ in Chapter I. (Direct substitution in (1.28) is of course impossible since $n^2 = 1$). The result is

$$q_m x = \frac{\cos 2\theta}{2\tau(1-\tau)^\beta} + \frac{\beta}{2}\int^\tau \frac{d\tau}{\tau(1-\tau)^{\beta+1}} \qquad (2.1)$$

$$q_m y = \frac{\sin 2\theta}{2\tau(1-\tau)^\beta} . \qquad (2.2)$$

By rewriting $\tau = \frac{q^2}{q_m^2}$ and letting $q_m \to \infty$, the corresponding incompressible flow is recovered as

$$x = \frac{\cos 2\theta}{2q^2}$$

$$y = \frac{\sin 2\theta}{2q^2}$$

with $$\Psi = \frac{\sin\theta}{q}$$

$$\phi = \frac{\cos\theta}{q} .$$

This corresponds to flow round a semi-infinite flat plate, the streamlines being parabolas (Fig. (i)).

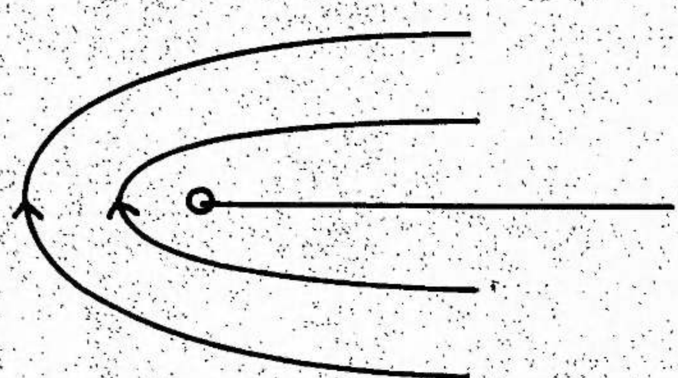


Fig. (i)

It is natural to assume that the general pattern of the compressible flow will be roughly equivalent. But one obvious difference will be the behaviour at the corner if it exists in this new flow. The velocity is infinite in the incompressible flow but this is impossible in the compressible flow where the velocity cannot exceed the finite limit q_m.

To begin with, the line $\Psi = 0$ in the compressible flow is considered. This is obtained by putting $\theta = 0$ and using (2.1) and (2.2). The line has coordinates in terms of the parameter τ given by

$$x = \frac{1}{q_m}\left\{\frac{1}{2\tau(1-\tau)^\beta} + \frac{\beta}{2}\int^\tau \frac{d\tau}{\tau(1-\tau)^{\beta+1}}\right\}$$

$$y = 0.$$

It is a simple matter to verify that as $\tau \to 0$, $x \to +\infty$ and that x decreases as τ increases up to the point for which $\tau = \frac{1}{2\beta+1} = \tau_s$ the local speed of sound. At this point $\frac{dx}{d\tau}$ changes sign, the streamline $\Psi = 0$ apparently turns back on itself and, as $\tau \to 1$, $x \to +\infty$ once more. This is quite different from the incompressible case where the streamline does turn sharply but requires infinite velocity to do so. It can be shown that there exists Ψ_0 such that all streamlines $\Psi < \Psi_0$ are cusped but that for $\Psi > \Psi_0$ the streamlines are smooth and behave roughly like the outer parabolas of Fig. (1).

The explanation lies in the behaviour of the various functions at the singularities of the transformation $u, v \to x, y$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(\phi,\Psi)}\,\frac{\partial(\phi,\Psi)}{\partial(q,\theta)}\,\frac{\partial(q,\theta)}{\partial(u,v)}$$
$$= \left(\frac{\rho_0}{\rho q^2}\right)^2 \left\{ (M^2-1)\Psi_\theta^2 - q^2\Psi_q^2 \right\}.$$

For $M < 1$ this expression can only vanish when $\Psi_q = \Psi_\theta = 0$. It can be shown that in this case the singularity is isolated. However for $M > 1$ the expression can vanish over some curve in both hodograph and physical planes. In the physical plane the curve is called a "limit line". In Ringleb's solution it is a double cusped line as shown in Fig. (ii), which is reproduced from (22). K denotes the sonic line and $\mathfrak{L}$ the limit line.

The limit line splits the x,y plane into two parts. To the right of it q, θ, ϕ and Ψ are three-valued functions of x and y: to the left they are single-valued. A streamline meeting the limit line becomes cusped and turns back provided it is on the same sheet in the physical plane as the limit line. That is, provided its values of q, θ are the same as those of the limit line at the same point. If not, the streamline goes through the limit line non-singularly. A family of non-singular streamlines can be picked out to give flow round a 180° bend but not including the limiting straight streamline as in the incompressible

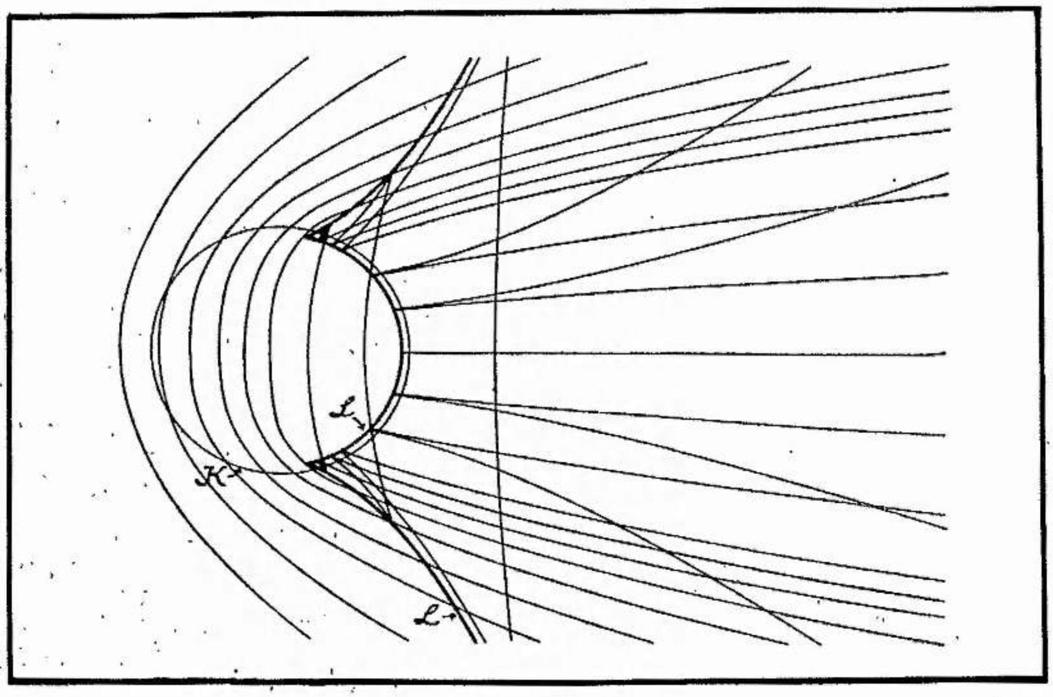


Fig. (ii)

[Reproduced by kind permission of the publishers from "Supersonic Flow and Shockwaves" by Courant and Friedrichs.]

case. The aggregate of these non-singular streamlines represents a physically possible continuous flow round a fixed boundary whose shape is determined by $\Psi = \Psi_0$. It is interesting to note that this flow contains a finite supersonic region embedded in the subsonic region and bounded in part by the wall and that the maximum Mach number attained is as high as 2.5. Thus although limit lines cannot occur in the subsonic region, the mere presence of a supersonic region is not sufficient to ensure the breakdown of the continuity of the flow.

The situation is a little clearer in the hodograph plane for x, y, ϕ and Ψ are single-valued functions of q and θ. The streamlines are circles touching the line $\upsilon = 0$ at O and the limit line (or rather the image of the limit line in the hodograph plane) is an ellipse (Fig.(iii)). The figure shows the existence of streamlines which are in part supersonic but which do not cut the limit line. The behaviour of the streamlines in the hodograph plane is of course non-singular as the irregularity in the physical plane is purely the result of the singularity in the transformation when returning to the x, y coordinates.

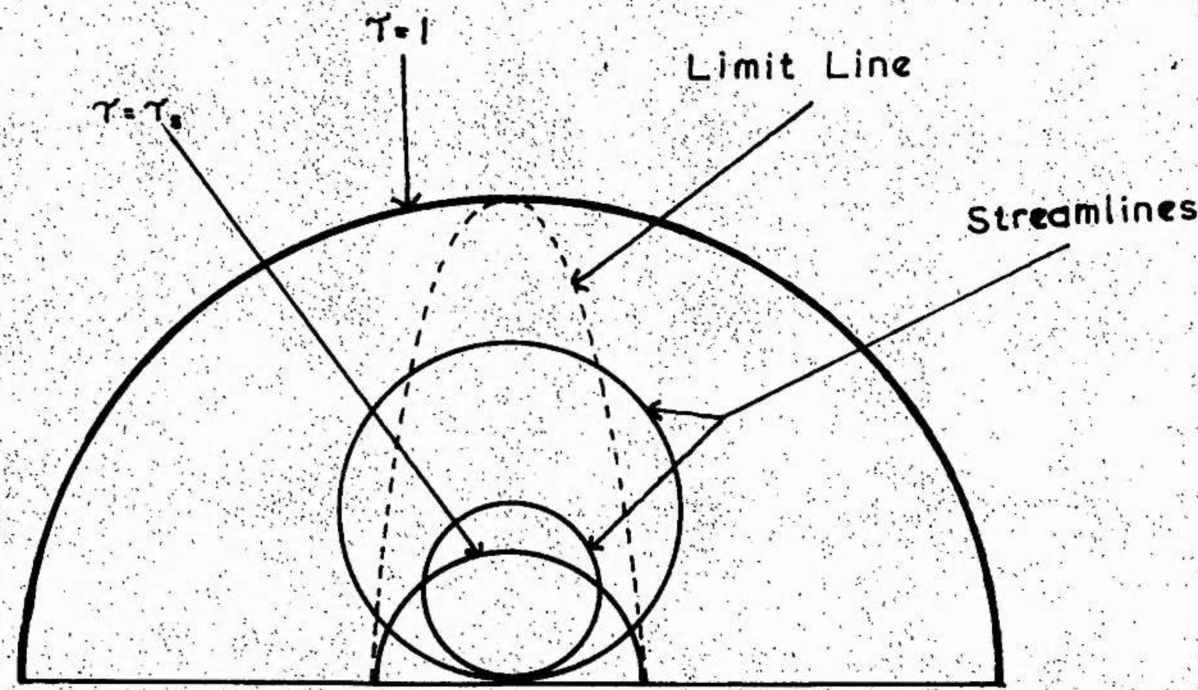


Fig. (iii).

Termination of the Hypergeometric Series. (ii) For the hypergeometric series to terminate it is obviously necessary for $(a_n + r)(b_n + r)$ to vanish for some positive integral value of r. That is

$$r^2 + r(n - \beta) - \frac{\beta n(n+1)}{2} = 0$$

For general β this requires first $r^2 + rn = 0$ which yields nothing new since n cannot be a negative integer < -1.

However for $\beta = 2.5$, the accepted value for air corresponding to $\gamma = 1.4$, the equation reduces to

$$5n^2 + n(5 - 4r) + 10r - 4r^2 = 0$$

For $r = 1,2$ the corresponding values of n are complex but for $r = 3$, $(5n + 3)(n - 2) = 0$.

Thus $\psi_2(\tau)$ and $\psi_{-3/5}(\tau)$ have only four terms each and these particular solutions can be examined in detail without heavy computation. The position coordinates are obtained by direct substitution in (1.28). The case $n = 2$ was described by Williams (10). It represents the flow in the interior of a right angle but it again is complicated by limit lines. The case $n = -\frac{3}{5}$ is of little physical interest and roughly speaking represents flow round an angle of 300° so that a two-sheeted surface in the physical plane has to be postulated (Fig. iv).

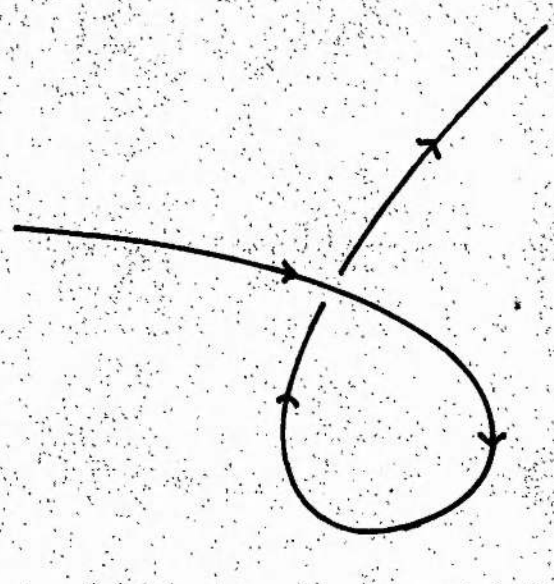


Fig. (iv).

There are no other terminating series with rational n corresponding to a value of r which is reasonably small.

<u>Generalization of Known Incompressible Flow.</u> It has already been pointed out that a given solution in the hodograph variables has to be solved before the corresponding problem in the physical plane is apparent. Nevertheless it seems that a good starting point might be to attempt to generalize known incompressible flows in the hope that the solutions might represent configurations in some way analogous to those of the original incompressible flows. As an example the classical flow round a circular cylinder of unit radius is considered.

This is given by

$$\phi = U(r + \tfrac{1}{r}) \cos\theta' \ , \quad \Psi = U(r - \tfrac{1}{r}) \sin\theta'$$

where U is the velocity at infinity. That is

$$w = U(z + \tfrac{1}{z}) \qquad \left(z = r e^{i\theta'} , \ w = \phi + i\Psi \right)$$

$$\frac{dw}{dz} = U(1 - \tfrac{1}{z^2}) = qe^{-i\theta}$$

Let $\frac{q}{U} e^{-i\theta} = \zeta$, Then $z = (1 - \zeta)^{-\frac{1}{2}}$ and

$$w = U\left\{ (1 - \zeta)^{-\frac{1}{2}} + (1 - \zeta)^{\frac{1}{2}} \right\}. \qquad (2.3)$$

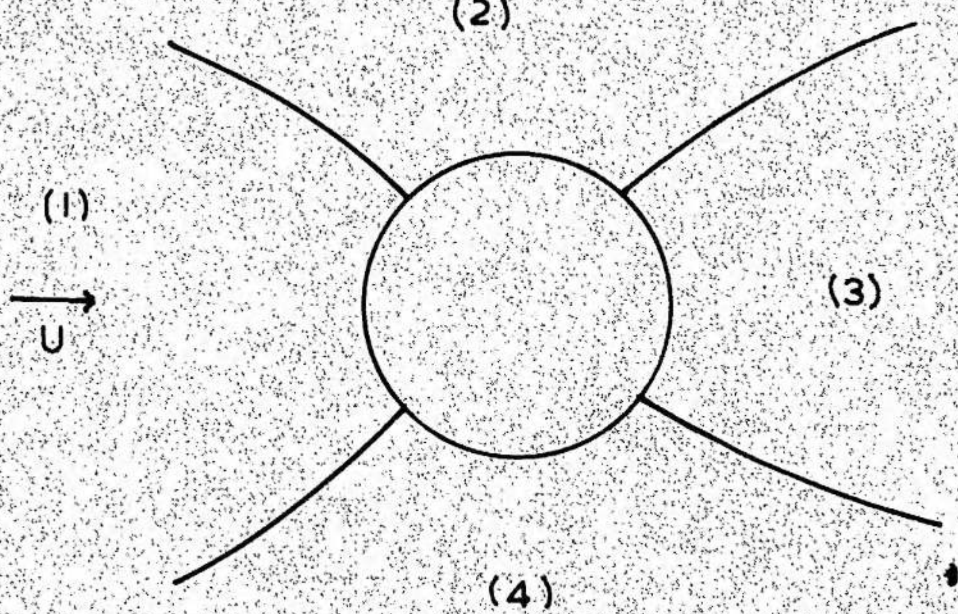


Fig. (v).

ω is now expanded in powers of ζ. The reason for this is that the expression $\frac{\psi_n(\tau)}{\psi_n(\tau_0)} e^{-in\theta}$ might be said to correspond to $\left(\frac{q}{U}\right)^n e^{-in\theta}$ and a power series in ζ can be replaced term by term by expressions such as this. The imaginary part of the resulting expression will certainly satisfy the equation for Ψ. Care must be taken however in selecting the particular expansion of (2.3), depending on the region of the physical plane to which it refers. Fig. (v) shows the physical plane split into four parts. In regions (1) and (3) $q < U$, that is $|\zeta| < 1$ and in

(2) and (4) $q > U$ and $|\zeta| > 1$. Moreover in (1) $z(\zeta) \to -1$ as $\zeta \to 0$ and in (3) $z(\zeta) \to +1$ as $\zeta \to 0$. Hence the appropriate expansion for (1) may be written as

$$\omega_{(1)} = -2U\left(1 + c_1\zeta + c_2\zeta^2 + \ldots\right) \tag{2.4}$$

$$\Psi_{(1)} = 2U\left(c_1 \frac{q}{U}\sin\theta + c_2\left(\frac{q}{U}\right)^2 \sin 2\theta + \ldots\right). \tag{2.5}$$

In region (2)

$$\omega_{(2)} = U\left(d_0\zeta^{\frac{1}{2}} + d_1\zeta^{-\frac{1}{2}} + d_2\zeta^{-\frac{3}{2}} + \ldots\right)$$

$$\Psi_{(2)} = U\left(-d_0\left(\frac{q}{U}\right)^{\frac{1}{2}}\sin\frac{\theta}{2} - d_1\left(\frac{q}{U}\right)^{-\frac{1}{2}}\sin\left(-\frac{\theta}{2}\right) - \ldots\right) \tag{2.6}$$

and $\Psi_{(3)} = -\Psi_{(1)}$, $\Psi_{(4)} = -\Psi_{(2)}$.

An attempt is now made to generalize (2.5). If τ_0 is the infinity velocity upstream in the compressible flow then the new Ψ function for the region corresponding to (1) in this flow might be written tentatively as

$$\Psi = 2U\left\{c_1\frac{\psi_1(\tau)}{\psi_1(\tau_0)}\sin\theta + c_2\frac{\psi_2(\tau)}{\psi_2(\tau_0)}\sin 2\theta + \ldots\right\}. \tag{2.7}$$

It will be shown later than this series does in fact converge for $\tau < \tau_0$ provided $\tau_0 < \tau_s$ and that at $-\infty$ it does tend to a uniform flow with velocity τ_0 . In what follows the infinity velocity τ_0 will always be assumed to be subsonic.

Equation (2.6) can be modified in the same way,

the typical term being $-U\,d_n\,\dfrac{\psi_{-n+\frac{1}{2}}(\tau)}{\psi_{-n+\frac{1}{2}}(\tau_0)}\,\sin\,\overline{-n+\tfrac{1}{2}}\,\theta$.
But here a very important point must be made which was not recognized in the early work in this field. Namely, that although the series (2.6) is the analytic continuation of (2.5) (as is obvious from (2.3)), it is not true that the series obtained by modifying (2.6) as indicated is the analytic continuation for $\tau > \tau_0$ of the series (2.7). The two series are both locally possible but do not join up to give an analytically continuous solution in the physical plane as in the corresponding incompressible problem. It is, however, possible to start with series (2.7) and to find its correct analytic continuation. The various solutions then join up to give a flow round a body although no longer a circular body. This process is carried out in (11). This is an example of the deformation of the boundary conditions through modifying the solution to allow for compressibility. The problem in the physical plane is altered and in a way which can only be determined a posteriori from the new solution.

The problem of the correct analytic continuation was tackled separately by Lighthill (14) and Cherry (15), but before the results of their work are set down it is necessary to examine in more detail the function $\psi_n(\tau)$ regarded especially as a function of n. A detailed description of the properties of this function was given by Lighthill (13) and the particular properties of interest in the present problem which he established are now quoted.

Properties of $\psi_n(\tau)$. It is first of all necessary to introduce a new variable s in the subsonic region defined by

$$\frac{ds}{d\tau} = \left\{ - \frac{4(\gamma-1)\tau^2(1-\tau)}{(\gamma+1)\tau - (\gamma-1)} \right\}^{-\frac{1}{2}}. \tag{2.8}$$

This reduces the equation for Ψ to

$$\frac{\partial^2\Psi}{\partial s^2} + \frac{\partial^2\Psi}{\partial\theta^2} = \frac{\partial\Psi}{\partial s} \times (\text{some function of } s).$$

(2.8) can be integrated to give

$$s = \sigma + \sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\left\{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma+1)(1-\tau)}\right\}} - \tanh^{-1}\sqrt{\left\{\frac{(\gamma-1)-(\gamma+1)\tau}{(\gamma-1)(1-\tau)}\right\}}, \tag{2.9}$$

where σ is the value of s corresponding to $\tau = \tau_s = \frac{\gamma-1}{\gamma+1}$ and given explicitly by

$$\sigma = -\sqrt{\frac{\gamma+1}{\gamma-1}}\tanh^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}} + \frac{1}{2}\log 2(\gamma-1).$$

For $\gamma = 1.4$ this is approximately equal to $-1.17344\ldots$

By inverting (2.9) for small values of τ it can be shown that as $\tau \to 0$, $s \to -\infty$ and $\tau \sim e^{2s}$. s is a monotonic function of τ in the region $0 < \tau < \tau_s$.

Regarded as a function of the complex variable n for some fixed τ such that $0 < \tau \leq 1$, $\psi_n(\tau)$ is an analytic function except for the values $n = -2, -3, \ldots$.

where it has simple poles, the residue at n = -m being $-m\, C_m\, \psi_m(\tau)$ where C_m is given by

$$C_m = \frac{\Gamma(a_m)\,\Gamma(m+1-b_m)}{\Gamma(a_m-m)\,\Gamma(1-b_m)(m!)^2} .$$

It is easily verified that this definition is symmetric in a_m and b_m and that $C_m > 0$ for the value of γ accepted for air.

By means of the asymptotic formula for the Γ function, C_m can be written as

$$C_m = \frac{1}{2\pi m} e^{-2\sigma m}\left(1 + O\left(\tfrac{1}{m}\right)\right) \quad \text{for large m.} \qquad (2.10).$$

Further, $\psi_n(\tau)$ can be expanded for $0 \leq \tau < \tau_s$ and for all complex n apart from n = -2, -3, ... as

$$\psi_n(\tau) = e^{ns}\left(1 + n \sum_{m=2}^{\infty} \frac{C_m e^{ms}\psi_m(\tau)}{n+m}\right). \qquad (2.11).$$

For large values of $|n|$ and away from the singularities

$$\psi_n(\tau) \sim e^{ns}\, V(\tau) \qquad (2.12).$$

for $0 < \tau < \tau_s$ where $V(\tau) = \left[\frac{(1-\tau)^{2\beta+1}}{1-\tau/\tau_s}\right]^{\frac{1}{4}}$

Upper limits for sonic and supersonic τ are given by $|\psi_n(\tau)| \leq B(\delta, \varepsilon)\, |n^{\frac{1}{6}}|\, e^{n\sigma}$ (2.13).

when $\tau_s \leq \tau \leq 1-\varepsilon$, n is real and $|n + m| \geq \delta$ for all non-negative integers m.

<u>Second Solution of the Differential Equation.</u> In view of the non-existence of $\psi_{-n}(\tau)$ for n = 2, 3, ... some other second solution must be found for such values of n.

Following Lighthill's notation this will be denoted by $\psi_n^*(\tau)$ and defined by

$$\psi_n^*(\tau) = \lim_{m \to -n} \left[\psi_m(\tau) - \frac{m C_n \psi_{-m}(\tau)}{m+n} \right]. \tag{2.14}$$

If the right hand side of (2.14) is substituted in the differential equation before taking the limit the resulting expression is

$$(n^2-m^2)\left[\psi_m(\tau) - \frac{m C_n \psi_{-m}(\tau)}{m+n} \right].$$

The term inside the square brackets is non-infinite as $m \to -n$ by the residue values of the poles of $\psi_m(\tau)$ or directly from (2.11). Hence $\psi_n^*(\tau)$ satisfies the differential equation and will be taken as the second fundamental solution.

(2.14) can be written as

$$\psi_n^*(\tau) = \lim_{m \to -n} \left[\psi_m(\tau) - \frac{m C_n \psi_n(\tau)}{m+n} + \frac{m C_n \left(\psi_n(\tau) - \psi_{-m}(\tau) \right)}{n+m} \right]$$

i.e. $$\psi_n^*(\tau) = \lim_{m \to -n} \left[\psi_m(\tau) - \frac{m C_n \psi_n(\tau)}{m+n} \right] - n C_n \frac{d}{dn}(\psi_n(\tau)). \tag{2.15}$$

For large n, $\psi_n^*(\tau)$ has the following asymptotic forms.

For $0 < \tau < \tau_s$, $\psi_n^*(\tau) \sim e^{-ns} V(\tau)$. (2.16)

For $\tau_s < \tau < 1$, $\psi_n^*(\tau) \sim |V(\tau)| e^{-n\sigma}\left\{\cos(nt+\frac{\pi}{4}) - \frac{2\sigma}{\pi}\sin(nt+\frac{\pi}{4})\right\}$, (2.17)

where t is a real function of τ defined in $\tau_s < \tau < 1$.

<u>Methods of Analytic Continuation</u>. The problem of analytic continuation is now reconsidered in the light of the information regarding $\psi_n(\tau)$. In the first place the compressible equivalent of $\left(\frac{q}{U}\right)^n e^{-in\theta}$ was tentatively taken as $\frac{\psi_n(\tau)}{\psi_n(\tau_0)} e^{-in\theta}$. From (2.12) it is clear that $\left|\frac{\psi_n(\tau)}{\psi_n(\tau_0)}\right| \sim \left|\frac{V(\tau)}{V(\tau_0)}\right| e^{n(s-s_0)}$ where $s_0 = s(\tau_0)$ and so (2.7) converges for all $\tau < \tau_0$. This result was stated earlier. However the function $\frac{\psi_n(\tau)}{\psi_n(\tau_0)} e^{-in\theta}$ is not a very suitable "equivalent" function to use since zeros of $\psi_n(\tau_0)$ give poles and this introduces additional terms in the analytic continuation. The requirements of such a function $f_n(\tau, \tau_0)\, e^{-in\theta}$ are

(i) it must satisfy the differential equation for Ψ

(ii) the series $\Sigma\, c_n e^{-in\theta} f_n(\tau, \tau_0)$ must converge for $\tau < \tau_0$ but not for $\tau > \tau_0$. This is to give τ_0 the significance of a free stream velocity.

(iii) As $\tau, \tau_0 \to 0$, $f_n(\tau, \tau_0) \sim \left(\frac{q}{U}\right)^n$ For $\tau = \frac{q^2}{q_m^2}$ and if q is kept fixed the effect of letting $q_m \to \infty$ is to let $\tau \to 0$, that is the compressibility effects are being eliminated. In so far as the solution is a general-

-ization from an incompressible flow, that incompressible flow must be recoverable from it by letting $q_m \to \infty$.

The simplest expression which satisfies these requirements and the one which will be used throughout the work which follows is $e^{-ns_0}\psi_n(\tau)\,e^{-in\theta}$. (i) is obviously satisfied. (ii) is satisfied since, as $n \to \infty$, $f_n(\tau, \tau_0) \sim V(\tau)\, e^{n(s-s_0)}$. And (iii) because as $\tau \to 0$, $\psi_n(\tau) \sim \tau^{\frac{n}{2}}$ and thus $f_n(\tau,\tau_0) \sim \tau_0^{-\frac{n}{2}}\tau^{\frac{n}{2}} = \left(\frac{q}{u}\right)^n$.

The choice of e^{-ns_0} simplifies the analytic continuation but it is still considerably more involved than in the incompressible case where the expression for Ψ is usually in closed form initially. The series $\sum c_n\psi_n(\tau)\,e^{-ns_0}e^{-in\theta}$ has to be expressed in a suitable form from which another series can be extracted to cover the case $\tau > \tau_0$. It is not necessary nor even usual for n to range through positive integral values. A more generalized expansion (such as will occur in the next chapter) is usually required.

A general solution of the problem was formulated by Lighthill (14). If $\zeta = \left(\frac{q}{u}\right)e^{-i\theta}$ and $\omega(\zeta)$ is the complex potential for the given incompressible flow problem, then the stream function for the generalized flow is given by

$$\Psi = \mathcal{I}\left[\omega\left(e^{s-s_0-i\theta}\right) + \sum_{m=1}^{\infty} C_m \psi_m(\tau)\, e^{m(s_0+i\theta)} \int_{\zeta_0}^{e^{s-s_0-i\theta}} \zeta^{m}\, d\omega(\zeta)\right] \qquad (2.18)$$

where $\mathcal{I}$ denotes the imaginary part of the following expression. Since s is only defined for subsonic values of τ, this expression as it stands cannot represent the continuation into the supersonic region. However it is possible to obtain a series expansion out of (2.18) which does not contain s explicitly and which converges for supersonic τ. The correct continuation of Ψ in the supersonic region is thus obtained.

An alternative method of analytic continuation was subsequently suggested by Lighthill, Goldstein and Craggs (11); it was based on the Barnes contour integral for the hypergeometric function. The method will not be described in detail here as it is used in a particular example in the next chapter. Briefly, however, the method is as follows. For a wide range of problems there is associated with the modified series for $\tau < \tau_0$ a certain function $F(\tau, \theta, \nu)$. The residues at those poles of $F(\tau, \theta, \nu)$ which lie in the right hand half of the complex ν plane generate the series term by term. The poles are all found to lie on the real axis. Accordingly the series may be represented by the integral of the function taken over the right hand infinite semi-circle together with the imaginary axis, suitably indented if necessary. For $\tau < \tau_0$ the integral over the semi-circle is found to vanish and the series is therefore represented by the integral of the function over the whole of the imaginary axis. This integral has to be shown to represent the stream function over an extended range. Hence it can be used to continue the series analytically. Moreover it is found that the integral of the function over the left hand semi-circle vanishes when $\tau_s > \tau > \tau_0$. Hence the analytic continuation of the series is determined by the new series generated by the residues of the poles of the function which lie in the left hand plane. These also fall on the real axis.

CHAPTER III.

THE WEDGE PROBLEM.

Statement of the Problem. The particular application in which the technique outlined in the last chapter is used is the flow past a wedge of finite sloping edge. That is to say, the original problem in the incompressible flow is that of flow past such a wedge. The bounding streamline is modified in the corresponding compressible flow but the resulting configuration is in many respects similar to that of the original problem.

The procedure adopted is now briefly summarized. The complex potential for an ideal fluid flowing past a wedge profile is obtained. It is then expanded (in the region where the velocity is less than the infinity velocity) in the velocity coordinates in a series similar to (2.4). This series is then generalized as described using the function e^{-ns_0}, and the analytic continuation of this series for $\tau > \tau_0$ is found. Finally the position coordinates for each part of the solution are obtained by means of (1.28) and the resulting expressions investigated.

Two problems are considered. First the flow past a wedge in a wind tunnel with straight walls, the wedge being placed in the centre of the tunnel. From this the flow past a wedge in an infinite stream can be deduced by letting the width of the tunnel tend to infinity but some care is required in obtaining the limit and the problem can be done independently by a direct application of the Schwarz-Christoffel transformation. In both cases the direction of the steady flow at infinity is parallel to the axis of the wedge. The term wind tunnel is used generally to mean the lateral boundary of the fluid.

<u>Ideal Flow past a Wedge in a Wind Tunnel</u>. The configuration for the first problem is shown in Fig. (vi).

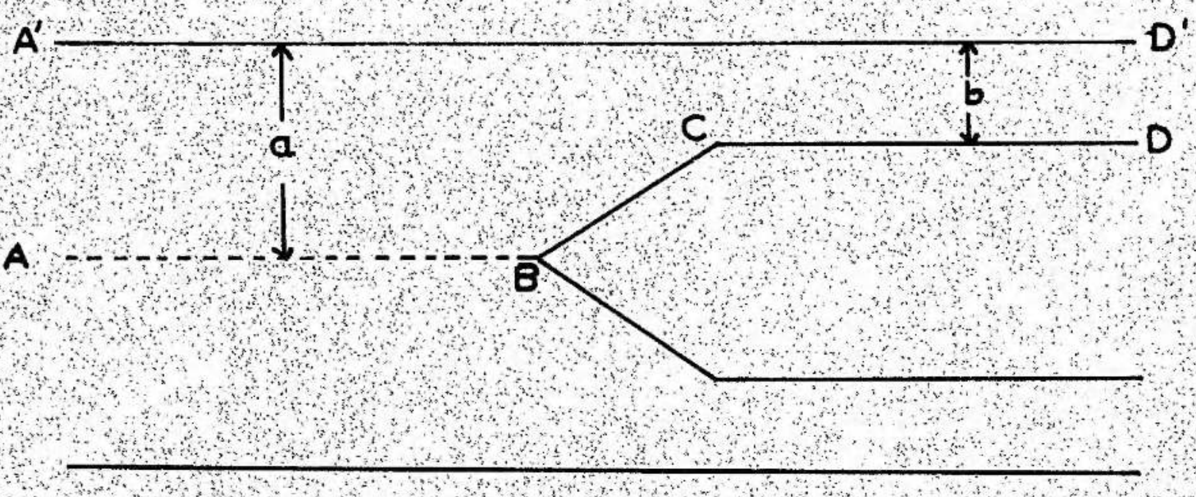


Fig.(vi)

By symmetry only the upper half of the figure need be considered, that is the region ABCDD'A'. The width of the tunnel is 2a, that of the wedge 2(a-b) and the semi-angle of the wedge is $\mu\pi$.

ABCD is the streamline $\Psi = 0$ and A'D' is $\Psi = Ua$, where U is the velocity at infinity upstream. By continuity, the velocity at infinity downstream is $\frac{a}{b}U = rU$ say.

Then $\phi(A) = -\infty$, $\phi(D) = +\infty$. $\phi(B)$ may be taken as zero. Let $\phi(C) = \alpha > 0$.

The ω plane is shown in Fig. (vii).

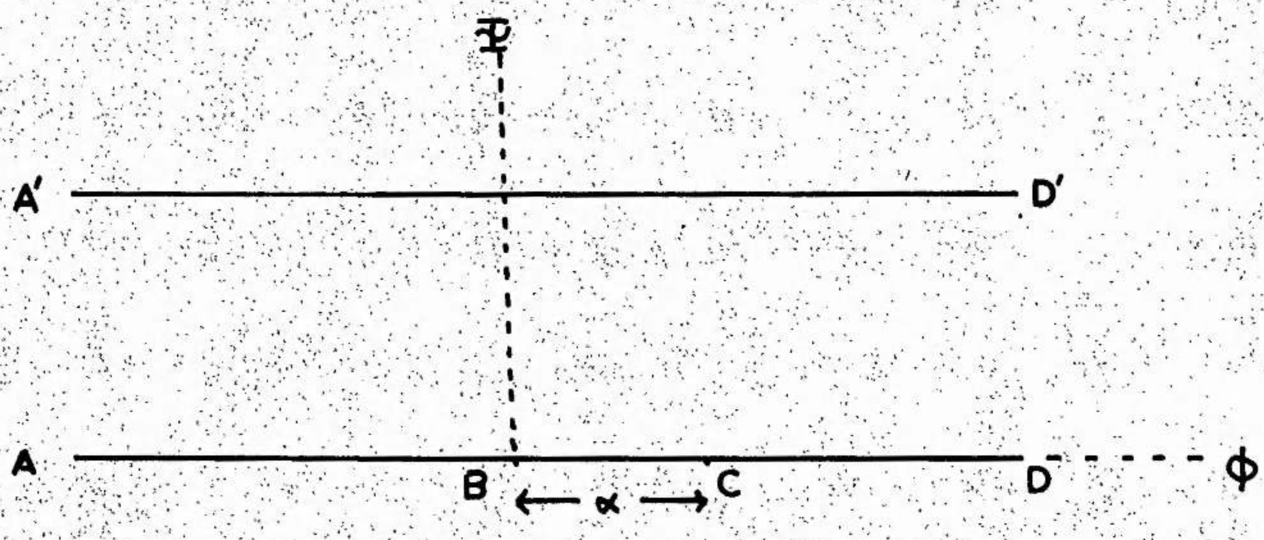


Fig.(vii)

The Q plane, where $Q = \log \zeta = \log \frac{q}{U} - i\theta$ is shown in Fig. (viii).

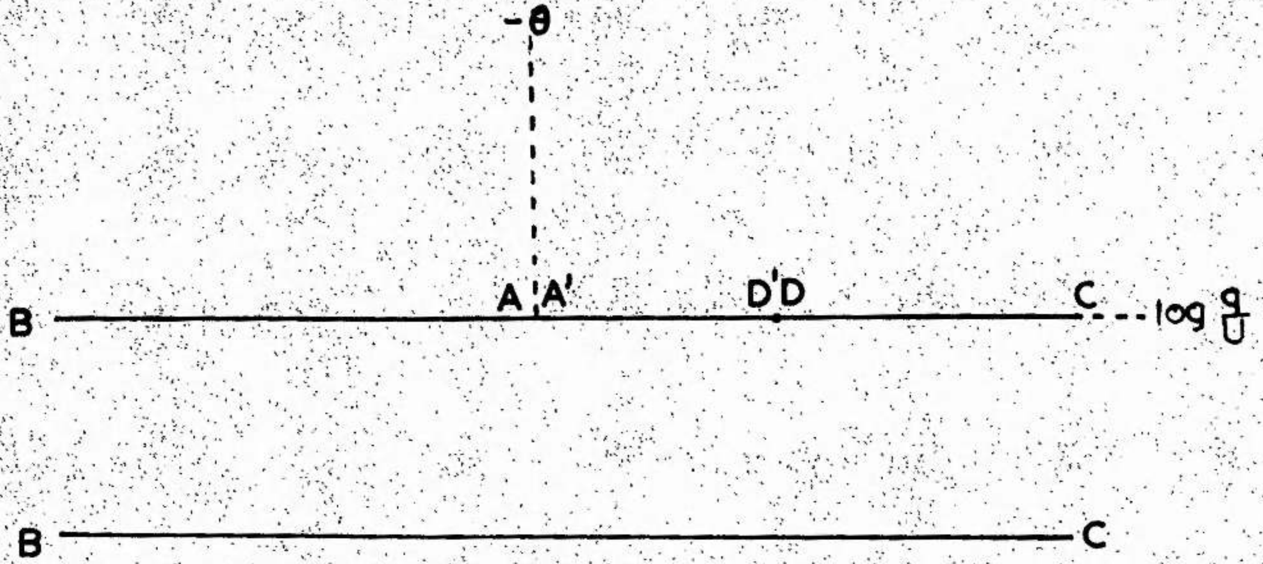


Fig. (viii)

The Schwarz-Christoffel transformation is now used to map the w and Q planes into one another. The appropriate domain of each is mapped onto the upper half of a t-plane in such a way that the real t axis corresponds to the contour ABCDD'A'A in each plane (Fig. (ix)). Three points can be chosen arbitrarily. They are chosen as follows :-

$$t(B) = 0$$
$$t(C) = 1$$
$$t(D,D') = \infty$$

Then $t(A,A') = -p$ say where $p > 0$.

Fig.(ix)

Required transformations are then given by

$$\frac{d\omega}{dt} = \frac{P}{t+p}$$
$$\frac{dQ}{dt} = \frac{S}{t(t-1)}.$$

Thus $$\omega = P \log (t+p) + P' \qquad (3.1)$$
$$Q = S \log \frac{t-1}{t} + S' \qquad (3.2)$$

To fit the appropriate constants in these expressions the following table is used.

	ω	Q	t
AA'	$-\infty\,(+Uai)$	0	$-p$
B	0	$-\infty\,(-i\mu\pi)$	0
C	α	$+\infty\,(-i\mu\pi)$	1
DD'	$+\infty\,(+Uai)$	$\log r$	$+\infty$

From conditions at D, D' $\qquad S' = \log r, \qquad P = \frac{Ua}{\pi}$

" " " C $\qquad S = -\mu \qquad P\log(1+p)+P' = \alpha$.

" " " B $\qquad P\log p+P' = 0$

" " " A, A' $\qquad S\log\frac{p+1}{p} + S' = 0$.

Hence

$$S = -\mu$$
$$S' = \log r$$
$$p = \frac{1}{r^{\frac{1}{\mu}}-1}$$
$$P = \frac{Ua}{\pi}$$
$$P' = \frac{Ua}{\pi}\log\left(r^{\frac{1}{\mu}}-1\right)$$
$$\alpha = \frac{Ua}{\pi}\log r^{\frac{1}{\mu}}.$$

Then (3.1), (3.2) become

$$\omega = \frac{Ua}{\pi}\log\left\{1+t\left(r^{\frac{1}{\mu}}-1\right)\right\} \tag{3.3}$$

$$Q = \log\left\{r\left(\frac{t}{t-1}\right)^{\mu}\right\}. \tag{3.4}$$

From (3.4) $\quad \zeta = r\left(\frac{t}{t-1}\right)^{\mu}$

or $\quad t = \frac{(\zeta/r)^{\frac{1}{\mu}}}{(\zeta/r)^{\frac{1}{\mu}}-1}$.

Hence finally, eliminating t,

$$\omega = \frac{Ua}{\pi}\log\frac{\zeta^{\frac{1}{\mu}}-1}{(\zeta/r)^{\frac{1}{\mu}}-1}. \tag{3.5}$$

The flow may be divided into three regions as shown in Fig. (x) in which $|\zeta| < 1$, $1 < |\zeta| < r$ and $|\zeta| > r$ in I, II and III respectively. Hence three different expansions are necessary, each of which is easily obtained by the ordinary logarithmic expansions of (3.5). There will, correspondingly, be three series expansions in the modified compressible flow. The extraction of the required series is however intricate and the properties of the flow will be considered by examination of the expression obtained by applying (2.18).

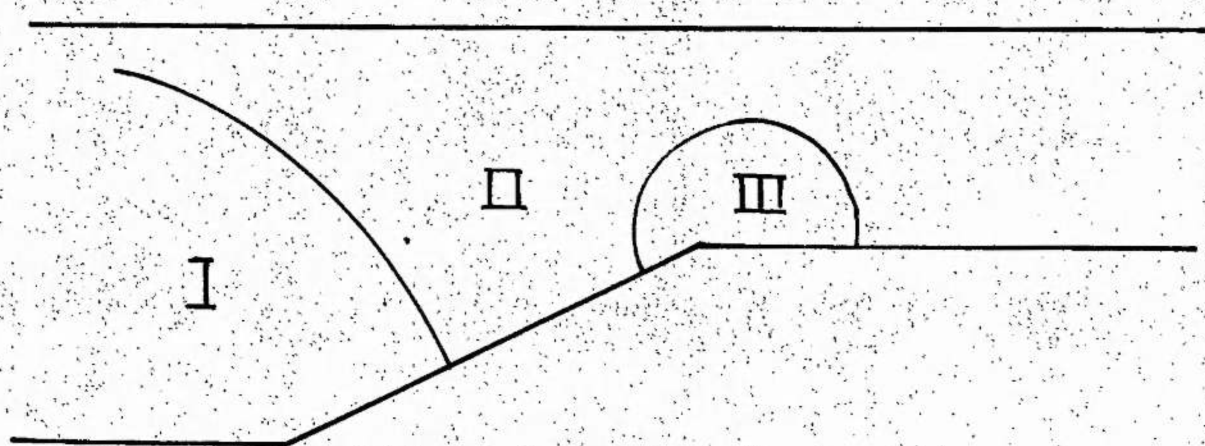


Fig.(x)

<u>Generalization of the Flow.</u> For the compressible flow, the complex stream function W (i.e. the function of which is the imaginary part) is given by

$$W = \omega(\lambda) + \sum_{m=2}^{\infty} C_m \psi_m(\tau)\, e^{m(s_0+i\theta)}\, I_m \qquad (3.6)$$

where $\lambda = e^{s-s_0-i\theta}$ and $I_m = \int_0^{\lambda} \zeta^{m} \frac{d\omega}{d\zeta}\, d\zeta$.

The lower limit of the integral is arbitrary in the general form but may be taken as zero here. This will be justified subsequently. $|\lambda|$ varies between 0 and $e^{\sigma-s_0}$ and arg λ between 0 and $-\mu\pi$.

$$\text{Now } I_m = \frac{U_a}{\mu\pi} \int_0^{\lambda} \zeta^{m} \left\{ \frac{\zeta^{\frac{1}{\mu}-1}}{\zeta^{\frac{1}{\mu}}-1} - \frac{\frac{1}{\tau}\left(\frac{\zeta}{\tau}\right)^{\frac{1}{\mu}-1}}{\left(\frac{\zeta}{\tau}\right)^{\frac{1}{\mu}}-1} \right\} d\zeta$$

$$= \frac{U_a}{\mu\pi}\left(\frac{1-\tau^{\frac{1}{\mu}}}{\tau^{\frac{1}{\mu}}}\right) \int_0^{\lambda} \frac{\zeta^{m+\frac{1}{\mu}-1}}{\left(\zeta^{\frac{1}{\mu}}-1\right)\left(\left(\frac{\zeta}{\tau}\right)^{\frac{1}{\mu}}-1\right)}\, d\zeta .$$

Put $\zeta = t^{\mu}$, $\tau = R^{\mu}$. Then

$$I_m = \frac{U_a}{\pi}(1-R) \int_0^{\lambda^{\frac{1}{\mu}}} \frac{t^{m\mu}}{(t-1)(t-R)}\, dt \qquad (3.7)$$

$$\text{and } W = \frac{U_a}{\pi}\left\{ \log \frac{1-\lambda^{\frac{1}{\mu}}}{1-\frac{1}{R}\lambda^{\frac{1}{\mu}}} + \sum_{m=2}^{\infty} C_m \psi_m(\tau)\, e^{m(s_0+i\theta)} (1-R) \int_0^{\lambda^{\frac{1}{\mu}}} \frac{t^{m\mu}}{(t-1)(t-R)}\, dt \right\}. \qquad (3.8)$$

The logarithmic term is made definite by giving it the value 0 as $\tau \to +0$ corresponding to $s \to -\infty$; and the value of $t^{m\mu}$ taken in the integrand is that which is real and positive when t is real.

<u>Discussion of the Solution.</u> Consider the value of W when $\tau < \tau_0$ and $\theta = 0$.

The logarithmic term is real since $\tau < \tau_0$ implies $s < s_0$ and since $R > 1$. For the same reasons all the integrals may be taken along the real axis and are therefore real. Hence W is real and consequently $\Psi = 0$ for $\theta = 0$, $\tau < \tau_0$. This is entirely analogous to the ideal flow.

Consider next $\theta = \mu\pi$. The logarithmic term is real for all values of s. In I_m put $t = ue^{-i\pi}$. This is the correct substitution because $\arg \zeta = -\mu\pi$ on $\theta = \mu\pi$ and $t = \zeta^{\frac{1}{\mu}}$.

Then $$I_m = -\frac{U_a}{\pi}(1-R)\int_0^{e^{\frac{s-s_0}{\mu}}} \frac{u^{m\mu}e^{-im\mu\pi}}{(u+1)(u+R)}\,du.$$

It follows from (3.6) that W is real for $\theta = \mu\pi$, $0 < \tau < \tau_s$; or that $\theta = \mu\pi$ is also a part of the streamline $\Psi = 0$, at any rate up to the speed of sound. Even if the continuation of the solution for supersonic τ were readily available, any interpretation of the result would be premature without considering the possibility of limit lines occurring. A limit line does in fact emanate from the sonic point of this straight streamline. For this reason no concrete deduction can be made about the region of flow corresponding in the original problem to the infinite straight side of the wedge.

The shape of the streamline corresponding to the wind tunnel wall is now considered. It is easy to see why this streamline in the ideal flow is straight. For let $\theta = 0$ and let $\frac{q}{u}$ increase through the value 1. Then since w has a logarithmic singularity at $\zeta = 1$, $\Psi = Ua$ for $\theta = 0$, $1 < |\zeta| < r$. By exactly the same argument $\psi_\omega(\lambda)$ increases by Ua as τ increases through the value τ_0. However, for values of $e^{\frac{s-s_0}{\tau}} > 1$, I_m is no longer real for $\theta = 0$. For the path of integration from 0 to $e^{\frac{s-s_0}{\tau}}$ has to be indented as shown in Fig. (xi) to get round the singularity at $t = 1$. The indentation must be made below $t = 1$ since arg ζ must be negative. The integrals over the straight parts of the path are real but the contribution from the indentation becomes, in the limit

$$\frac{Ua}{\pi} \times (1-R)\,\pi i \times \frac{1}{1-R} = Uai.$$

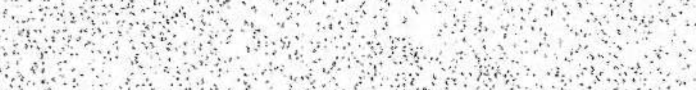


Fig.(xi)

The variation of the coefficients $C_m \psi_m(\tau) e^{ms_0}$ in (3.6) disposes of the possibility that Ψ is a non-zero constant for $\theta = 0$ and for $\tau > \tau_0$. This is another instance of the boundary problem being altered through the modification of the flow. The inherent interest of this particular problem is now lost since the flow is no longer contained in a wind tunnel with straight walls and there is little point in persevering with the additional complexity arising from the presence of the straight walls in the incompressible problem. Accordingly the rest of the work will be devoted to examining the flow which arises as the modification of the problem of an ideal fluid, which is unbounded laterally, flowing past a finite wedge.

Wedge in Infinite Stream. Such a problem can be deduced from the example just given by letting $r \to 1$. Immediate substitution however results in an indeterminacy and the required complex potential can be obtained ab initio by a direct application of the Schwarz-Christoffel transformation to the flow plane.

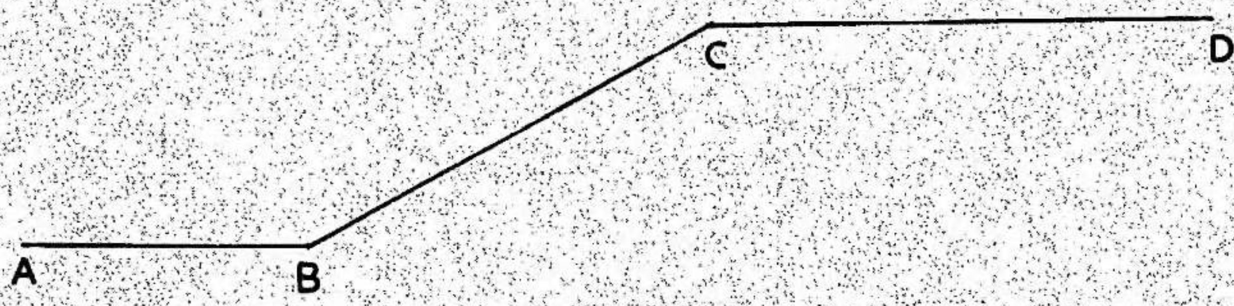


Fig.(xii)

Fig. (xii) shows the upper half of the flow field. This region, that is the region above ABCD, can be mapped into the upper half of the t-plane. The angle at B is $\pi - \mu\pi$ and at C is $\pi + \mu\pi$. Consequently the required transformation is given by

$$\frac{dz}{dt} = \frac{A(t-\alpha)^{\mu}}{t^{\mu}} \quad \text{and } A = 1 \text{ if } z \sim t \text{ at infinity.}$$

The points B, C, D in the figure correspond to $t = 0, \alpha, \infty$ respectively where $\alpha > 0$.

Now in the t-plane $\omega = Ut$ represents uniform flow. Hence the corresponding flow in the z-plane is given by $\quad \mathcal{U}\,\frac{dz}{dw} = \left(\frac{w - \mathcal{U}\alpha}{w}\right)^{\mu}$

Hence $\quad \zeta = \left(\frac{w}{w - \mathcal{U}\alpha}\right)^{\mu}$ where $\zeta = \frac{q}{\mathcal{U}}e^{-i\theta}$ as before.

$$\therefore \quad w = \frac{\alpha\,\mathcal{U}\,\zeta^{\frac{1}{\mu}}}{\zeta^{\frac{1}{\mu}} - 1}.$$

$\alpha U > 0$ and this may be rewritten

$$w = -\frac{c^2\,\zeta^{\frac{1}{\mu}}}{1 - \zeta^{\frac{1}{\mu}}}. \tag{3.9}$$

The basic features of the flow are quickly verified. The streamline $\Psi = 0$ is identified as corresponding to

$$\theta = 0 \quad \text{for} \quad U > q \geqslant 0$$

$$\theta = \mu\pi \quad \text{for} \quad 0 \leqslant q < \infty$$

$$\theta = 0 \quad \text{for} \quad \infty > q > U.$$

The series expansion for (3.9) in the region $q < U$ is given by

$$\omega_1 = -c^2 \sum_{n=1}^{\infty} \zeta^{\frac{n}{\mu}}. \tag{3.10}$$

The complex stream function for the modified flow is then

$$W_1 = -c^2 \sum_{n=1}^{\infty} e^{-\frac{n s_0}{\mu}}\,\psi_{\frac{n}{\mu}}(\tau)\,e^{-\frac{i n\theta}{\mu}}. \tag{3.11}$$

This gives $\quad \Psi_1 = c^2 \sum_{n=1}^{\infty} e^{-\frac{n s_0}{\mu}}\,\psi_{\frac{n}{\mu}}(\tau)\sin\frac{n\theta}{\mu}. \qquad (3.12)$

In Lighthill's general form this may be written as

$$W = \omega(\lambda) + \sum_{m=2}^{\infty} C_m\,\psi_m(\tau)\,e^{m(s_0 + i\theta)}\,I_m$$

where
$$I_m = \int_0^\lambda \zeta^m \frac{dw}{d\zeta}\, d\zeta \qquad \left(\lambda = e^{s-s_0-i\theta}\right)$$
$$= -\frac{c^2}{\mu}\int_0^\lambda \frac{\zeta^{m+\frac{1}{\mu}-1}}{(1-\zeta^{\frac{1}{\mu}})^2}\, d\zeta$$
$$= -c^2 \int_0^{\lambda^{\frac{1}{\mu}}} \frac{t^{m\mu}}{(1-t)^2}\, dt$$

Thus
$$W = -c^2\left[\frac{\lambda^{\frac{1}{\mu}}}{1-\lambda^{\frac{1}{\mu}}} + \sum_{m=2}^{\infty} C_m \psi_m(\tau) e^{m(s_0+i\theta)} \int_0^{\lambda^{\frac{1}{\mu}}} \frac{t^{m\mu}}{(1-t)^2}\, dt\right] \quad (3.13)$$

Although this is not the method actually used to determine the full continuation of W_1, it is of some interest to see how the original series (3.11) arises out of (3.13). It also serves the purpose of justifying the choice of $\zeta_0 = 0$ in I_m in (3.6). The procedure there is similar but the algebra here is somewhat less complicated.

If $s < s_0$, there exists T such that $|t| < T < 1$ in the integrand of I_m. Consequently the integrand can be expanded and then integrated term by term.

This gives $I_m = -c^2 \sum_{n=1}^{\infty} \frac{n\lambda^{n+m\mu}}{n+m\mu}$

$$W = -c^2\left[\sum_{n=1}^{\infty} \lambda^{\frac{n}{\mu}} + \sum_{m=2}^{\infty} C_m \psi_m(\tau) e^{m(s_0+i\theta)} \sum_{n=1}^{\infty} \frac{n\lambda^{m+\frac{n}{\mu}}}{n+m\mu}\right]. \quad (3.14)$$

Now for any fixed value of n, $\left| C_m \psi_m(\tau) e^{m(s_0+i\theta)} \frac{n\lambda^{m+\frac{n}{\mu}}}{n+m\mu}\right| < K(\tau) e^{m(-2\sigma+s+s_0)}$ (by the asymptotic properties of C_m, $\psi_m(\tau)$) and both series in n are absolutely convergent. Therefore the order of summation may be interchanged.

$$\therefore \quad W = -c^2 \sum_{n=1}^{\infty} \lambda^{\frac{n}{\mu}} \left\{ 1 + \sum_{m=2}^{\infty} \frac{\frac{n}{\mu} C_m \psi_m(\tau) e^{ms}}{\frac{n}{\mu}+m} \right\}$$

$$= -c^2 \sum_{n=1}^{\infty} e^{-\frac{ns_0}{\mu}} \psi_{\frac{n}{\mu}}(\tau)\, e^{-\frac{in\theta}{\mu}} \qquad \text{from (2.11).}$$

In this way the series (3.11) is recovered.

<u>Continuation of the Solution.</u> It is now required to continue the series analytically into the region $\tau > \tau_0$. The method indicated at the end of Chapter II will be adopted. First of all it is necessary to express (3.11) in closed form.

Consider the function $F(\tau, \theta, t) = \frac{\mu c^2}{2i} \frac{\psi_t(\tau)\, e^{-t(s_0+i\theta)+\pi i \mu t}}{\sin \mu\pi t}$ in the complex t-plane. In the right hand half of the t-plane, $F(\tau, \theta, t)$ has simple poles at $t = \frac{n}{\mu}$ for $n = 1, 2, \ldots$. The residue at $t = \frac{n}{\mu}$ is

$$\frac{c^2}{2\pi i}\, e^{-\frac{ns_0}{\mu}} \psi_{\frac{n}{\mu}}(\tau)\, e^{-\frac{in\theta}{\mu}}.$$

Now this is $\frac{1}{2\pi i}$ times the n^{th} term of (3.11) apart from a change in sign. Consequently, (3.11) can be written as

$$W_1 = \lim_{n\to\infty} \oint_{C_n} F(\tau, \theta, t)\, dt,$$

where C_n is the contour formed by the semi-circle centre O, radius $\frac{n+\frac{1}{2}}{\mu}$ and that part of the imaginary axis which forms its base, indented at O as shown, the whole contour being such that the real part of $t \geq 0$ everywhere on it (Fig. (xiii)). The contour is described in the clockwise direction.

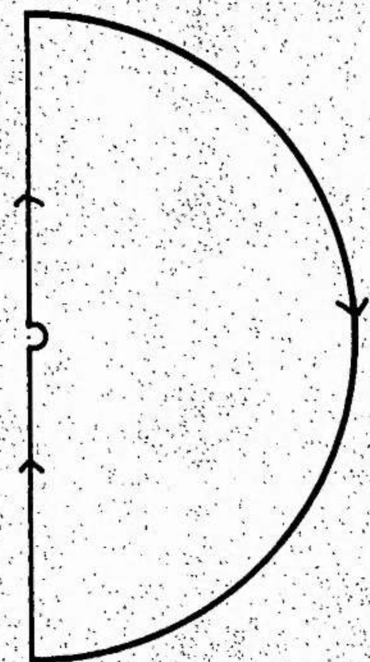


Fig.(xiii)

Consider the integral round the semi-circle. Put $t = Re^{i\phi}$ where $R = \frac{n+\frac{1}{2}}{\mu}$. Then on the semi-circle

$$\left|\int F(\tau,\theta,t)\,dt\right| \le A\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{e^{R(\cos\phi+i\sin\phi)s}\; e^{R(\cos\phi+i\sin\phi)(i\mu\pi-i\theta-s_0)}}{\sin\{\mu\pi(R\cos\phi+iR\sin\phi)\}}\right| R\,d\phi$$

where A is a constant independent of τ when $0 \le \tau \le \tau_s - \varepsilon$ by the asymptotic formula for $\psi_t(\tau)$.

$$\therefore \quad \left|\int F(\tau,\theta,t)\,dt\right| \le A\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{R\cos\phi(s-s_0)}\; e^{-R\sin\phi(\mu\pi-\theta)}}{\left|\sin\{\mu\pi(R\cos\phi+iR\sin\phi)\}\right|}\, R\,d\phi .$$

Consider a small interval $(-\eta, \eta)$ in the range of integration of ϕ. Then it is easily seen that, provided $|\mu\pi - \theta| < \mu\pi$, $\dfrac{e^{-R\sin\phi(\mu\pi-\theta)}}{|\sin\{\mu\pi(R\cos\phi + iR\sin\phi)\}|} < e^{-kR}$ for some fixed $k > 0$ outside the interval and that it is bounded within the interval.

Now if $s < s_0$ (corresponding to $\tau < \tau_0$) $e^{R\cos\phi(s-s_0)} \leq 1$. It is clear now that the integral tends to zero as $R \to \infty$, because of the term e^{-kR} in the ranges $\left(-\frac{\pi}{2}, -\eta\right)$ and $\left(\eta, \frac{\pi}{2}\right)$ and because of the term $e^{R\cos\phi(s-s_0)}$ in $(-\eta, \eta)$.

It is worth while noting here that the integral of the function taken over the left hand semi-circle vanishes when $\tau_0 < \tau < \tau_s$. For then $s - s_0 > 0$ but $\cos\phi \leq 0$. The proof in this case is practically identical. The radius R of the semi-circle has to be adjusted so as to keep both $\psi_t(\tau)$ and $\dfrac{1}{\sin\mu\pi t}$ bounded near the real axis.

Consider now $I = \int_{-i\infty}^{i\infty} F(\tau, \theta, t)\, dt$ indented at 0. If $t = iu$, then for large $|u|$

$$|F(\tau,\theta,t)| < B\left|\frac{e^{u(\theta-\mu\pi)}}{e^{\mu u\pi} - e^{-\mu u\pi}}\right| \quad \text{for} \quad 0 \leq \tau \leq \tau_s - \varepsilon$$

where B is a constant.

The integral consequently converges provided $|\theta - \mu\pi| < \mu\pi$ i.e. for $0 < \theta < 2\mu\pi$.

Hence I converges absolutely and uniformly for

$0 \leq \tau \leq \tau_s - \varepsilon$, $\delta \leq \theta \leq 2\mu\pi - \delta$.

By letting $R \to \infty$,

$$W_1 = -c^2 \sum_{n=1}^{\infty} e^{-\frac{ns_0}{\mu}} \psi_{\frac{n}{\mu}}(\tau) e^{-\frac{in\theta}{\mu}} = I.$$

It has now to be shown that the integral I represents the continuation of the flow into the region $\tau_1 < \tau < \tau_s$. To do this it is necessary to show that the integral satisfies the differential equation for the stream function, and that it is a continuous function of τ and θ with continuous partial derivatives of all orders at every point τ, θ at which it is required to represent the flow. Before this is done, however, an auxiliary result about the asymptotic form of $\psi_n'(\tau)$ for large values of n is required.

<u>Asymptotic Form of $\psi_n'(\tau)$</u>. The general form, given by Lighthill (13), for $\psi_n(\tau)$ is

$$e^{-ns}\psi_n(\tau) = \left[L(\tau, s_1) + \cdots + \left(-\tfrac{1}{n}\right)^r \frac{\partial^r L}{\partial s_1{}^r}\right]_{s_1 = s} + n^{-r} \sum_{m=2}^{\infty} \frac{(-m)^{r+1}}{n+m} C_m e^{ms} \psi_m(\tau) \quad (3.15)$$

where $\dfrac{\partial^r L}{\partial s_1{}^r} = \sum_{m=1}^{\infty} m^r C_m e^{ms_1} \psi_m(\tau)$.

The infinite series in (3.15) converges uniformly in $0 \leq \tau \leq \tau_s - \varepsilon$ by the asymptotic properties of $C_m \psi_m(\tau)$, and satisfied the necessary conditions for $e^{-ns}\psi_n(\tau)$ to have an asymptotic expansion in powers of $\frac{1}{n}$.

Differentiation term by term of (3.15) is valid if the resulting series is uniformly convergent.

Consider $e^{ms}\psi_m(\tau)$.

e^{2s} is an analytic function of τ in $0 \leq \tau \leq \tau_s - \frac{\varepsilon}{2}$ say, with a power series

$e^{2s} = \tau + \ldots\ldots$. This is proved in (13).

$\therefore \; e^{ms} = \tau^{\frac{m}{2}}\left\{1 + a_1\tau + \ldots\ldots\right\}$ convergent in $0 \leq \tau \leq \tau_s - \frac{\varepsilon}{2}$.

Also $\psi_m(\tau) = \tau^{\frac{m}{2}}\left\{1 + b_1\tau + \ldots\right\}$ convergent in $0 \leq \tau < 1$.

$\therefore \; \frac{(-m)^{r+1}}{n+m} C_m e^{ms}\psi_m(\tau) = \sum_{p=0}^{\infty} c_{pm}\tau^p$ where $c_{im} = 0$ $(i = 0,1,2,\ldots m-1)$ since multiplication of power series is valid.

$\therefore \; \sum_{m=2}^{\infty} \frac{(-m)^{r+1}}{n+m} C_m e^{ms}\psi_m(\tau) = \sum_{m=2}^{\infty}\sum_{p=0}^{\infty} c_{pm}\tau^p$ and this double series is uniformly convergent in $0 \leq \tau \leq \tau_s - \varepsilon$.

$\therefore \; \sum_{m=2}^{\infty} c_{pm}$ converges to a limit d_p say and $\sum_{p=0}^{\infty} d_p\tau^p$ is convergent in $0 \leq \tau \leq \tau_s - \varepsilon$. Hence, being a power series, it is uniformly and absolutely convergent in $0 \leq \tau \leq \tau_s - 2\varepsilon$.

Now $\frac{d}{d\tau}\left(e^{ms}\psi_m(\tau)\right) = \frac{1}{\tau}\sum_{p=0}^{\infty} p\, c_{pm}\tau^p$ (since $c_{om} = 0$) and the resulting series certainly converges absolutely in $0 \leq \tau \leq \tau_s - 2\varepsilon$.

It is required to show that $\sum_{m=2}^{\infty}\sum_{p=0}^{\infty} p\, c_{pm}\tau^p$ converges uniformly in $0 \leq \tau \leq \tau_s - 2\varepsilon$.

Now $\sum_{m=2}^{\infty} c_{pm} = d_p$ where $\sum_{p=0}^{\infty} d_p \tau^p$ converges in $0 \leq \tau \leq \tau_s - \varepsilon$.

$\therefore \sum_{m=2}^{\infty} p c_{pm} = p d_p$ and $\sum_{p=0}^{\infty} p d_p \tau^p$ converges absolutely in $0 \leq \tau \leq \tau_s - 2\varepsilon$ since the derivative of a power series has the same radius of convergence as the original. Hence $\sum_{m=2}^{\infty} \frac{(-m)^{r+1}}{n+m} C_m \frac{d}{d\tau}\left(e^{ms}\psi_m(\tau)\right)$ converges uniformly in $0 \leq \tau \leq \tau_s - 2\varepsilon$ and the derivative of (3.15) satisfies conditions for an asymptotic series.

Hence $e^{-ns}\left\{\psi_n'(\tau) - n\frac{ds}{d\tau}\psi_n(\tau)\right\} = A(\tau) + \frac{B(\tau)}{n} + \cdots$

i.e. $\psi_n'(\tau) = n\frac{ds}{d\tau} e^{ns} V(\tau)\left\{1 + O\left(\frac{1}{n}\right)\right\}$. (3.16)

Similarly $\psi_n''(\tau) = n^2\left(\frac{ds}{d\tau}\right)^2 V(\tau)\left\{1 + O\left(\frac{1}{n}\right)\right\}$

Now consider $I = \int_{-i\infty}^{i\infty} F(\tau,\theta,t)\,dt$, in the range $0 \leq \tau \leq \tau_s - 2\varepsilon$, $\delta \leq \theta \leq 2\pi - \delta$.

F is a continuous function of τ, θ and t and I is uniformly convergent. Therefore I is continuous.

Also $\int_{-i\infty}^{i\infty} \frac{\partial F(\tau,\theta,t)}{\partial \tau}\,dt$ is a uniformly convergent integral and the integrand is a continuous function of τ and θ. Similarly $\frac{\partial^2 I}{\partial \tau^2}$, $\frac{\partial I}{\partial \theta}$ and $\frac{\partial^2 I}{\partial \theta^2}$ are continuous functions of τ, θ and are obtained explicitly by differentiating under the integral sign. It follows immediately by formal substitution that I satisfies (1.23) and consequently that $\mathcal{J} I$ does so. The continuity of all higher derivatives of I can be proved by the same method.

$\mathcal{J}$ I therefore represents the stream function for the flow in the region $0 \leq r \leq r_s - 2\varepsilon$, $\delta \leq \theta \leq 2\mu\pi - \delta$. It has already been shown that the integral of the function taken round the left hand semi-circle vanishes when $r_s > r > r_0$ as the radius R tends to infinity through suitable values. It follows that the complex stream function in the range $r_0 < r \leq r_s - 2\varepsilon$ is obtained by evaluating the series which are generated by the residues at the poles of the function which lie in the left hand part of the plane, together with the pole at the origin. These series are now obtained.

No stipulation has yet been made as to the nature of μ. To give a physically reasonable flow it will be assumed that $0 < \mu < \frac{1}{2}$. It is observed that $\psi_t(r)$ has poles at $t = -2, -3, \ldots$ and that $\frac{1}{\sin \mu\pi t}$ has poles at $t = 0, -\frac{1}{\mu}, -\frac{2}{\mu}, \ldots$ If μ is non-rational these poles are all separate and the function $F(r, \theta, t)$ has only simple poles. But if μ is rational, the singularities will coalesce to give an infinity of double poles together with another sequence of simple poles. The two cases have to be examined separately. The simpler case when μ is non-rational is considered first.

<u>Continuation for non-Rational μ.</u> The contribution to the series made by the pole of $\psi_t(\Upsilon)$ at $t = -m$ is

$$\frac{c^2 \mu\pi m C_m \psi_m(\Upsilon) e^{m(s_0+i\theta-i\mu\pi)}}{\sin m\mu\pi}$$

The contribution from the pole at $t = \frac{n}{u}$ is

$$c^2 \psi_{-\frac{n}{\mu}}(\Upsilon)\, e^{\frac{n}{\mu}(s_0+i\theta)}$$

The pole at 0 contributes c^2. Hence

$$W_2 = \lim_{R\to\infty} c^2\left[1 + \sum_{n} \psi_{-\frac{n}{\mu}}(\Upsilon) e^{\frac{n}{\mu}(s_0+i\theta)} + \mu\pi \sum_{m} \frac{m C_m \psi_m(\Upsilon) e^{m(s_0+i\theta-i\mu\pi)}}{\sin m\mu\pi}\right] \quad (3.17)$$

where $\sum_{n}$, $\sum_{m}$ denote the sums of the terms arising from the poles which occur in the semi-circle of radius R. Care must be taken not to identify the expressions immediately with the sum of two infinite series. However the order may be rearranged and W_2 written as

$$W_2 = c^2\left[1 + \sum_{n=1}^{\infty} \psi_{-\frac{n}{\mu}}(\Upsilon) e^{\frac{n}{\mu}(s_0+i\theta)} + \mu\pi \sum_{m=2}^{\infty} \frac{m C_m \psi_m(\Upsilon) e^{m(s_0+i\theta-i\mu\pi)}}{\sin m\mu\pi}\right] \quad (3.18)$$

if the resulting series are absolutely convergent.

The reason for the difficulty arising here is that $\psi_{-\frac{n}{\mu}}(\Upsilon)$ can be made arbitrarily large by taking n large enough in such a way as to ensure that $\frac{n}{\mu}$ is sufficiently close to a positive integer k; for $\psi_{-k}(\Upsilon)$ is infinite at $k = 2, 3, \ldots\ldots$ Similarly $\frac{1}{\sin m\mu\pi}$ can be made arbitrarily large. It might happen, therefore, that although the combination of the two gives rise to the convergent sequence (3.17) for W_2 (as it must by examination of the contour integral provided R is suitably chosen) it might not be possible to rewrite this as (3.18).

However Pack has shown, in an unpublished note, that W_2 is in fact absolutely convergent for almost all values of μ (i.e. for all μ except a set of measure zero) as a consequence of a theorem in the Theory of Numbers. Moreover the resulting series converge absolutely for all τ such that $\tau_0 < \tau < 1$ and for all θ. It follows that the series provide the continuation of the stream function into these regions, thus removing the restriction $\tau < \tau_s$ <u>and all restrictions on θ, in particular $\theta \neq 0$</u>. This is true for almost all values of μ. Although this means in theory that an enumerable set of μ have to be excluded, in practice, for any given wedge angle $\mu\pi$, μ' can be found arbitrarily near to μ and not belonging to the null set. Thus, by physical continuity, the solution for a given wedge angle can be approximated to, to any required degree of accuracy. However it will be assumed in what follows that μ does not belong to the null set and therefore (3.18) is the correct representation of W_2 and Ψ_2 is given by

$$\Psi_2 = c^2\left[\sum_{n=1}^{\infty} \psi_{-\frac{n}{\mu}}(\tau)\, e^{\frac{ns_0}{\mu}} \sin\frac{n\theta}{\mu} + \mu\pi \sum_{m=2}^{\infty} \frac{m C_m \psi_m(\tau) e^{ms_0} \sin m(\theta-\mu\pi)}{\sin m\mu\pi}\right] \quad (3.19)$$

It is obvious that the series (3.19) is badly adapted to a numerical analysis of the flow. However, by means of (3.19) and the corresponding series (3.12), the type of flow obtained can be examined qualitatively. (3.12) is more easily adapted to computation and some numerical results have been obtained. The continuation of (3.12) when μ is rational, obtained by evaluating the residues at the double poles, is considerably more complicated but it exhibits the same general features as (3.19). The continuation for μ rational will be found and this, together with (3.19), will be used to illustrate the properties of Ψ_2.

Consider first, however, (3.12), the series appropriate to $\tau < \tau_0$. Using the incompressible solution as a guide, this might be expected to cover a region in the physical plane extending from $-\infty$ to a line which meets the sloping edge of the wedge. This will be shown to be the case.

In the first place, $\Psi = 0$ for $\theta = 0$ or $\theta = \mu\pi$ for all τ for which the series converges. The position coordinates are obtained by means of (1.28).

$$Z = E(\tau)\sum_{n=1}^{\infty}\left[\frac{e^{(\frac{n}{\mu}+1)i\theta-\frac{n\xi_0}{\mu}}}{\frac{n}{\mu}+1}\left\{\frac{n}{2\mu}\psi_{\frac{n}{\mu}}(\tau)-\tau\psi_{\frac{n}{\mu}}'(\tau)\right\}-\frac{e^{-(\frac{n}{\mu}-1)i\theta-\frac{n\xi_0}{\mu}}}{\frac{n}{\mu}-1}\times\right.$$
$$\left.\left\{\frac{n}{2\mu}\psi_{\frac{n}{\mu}}(\tau)+\tau\psi_{\frac{n}{\mu}}'(\tau)\right\}\right] \qquad (3.20)$$

where $E(\tau) = \dfrac{c^2}{q_m\tau^{\frac{1}{2}}(1-\tau)^{\beta}}$.

It is easily seen that (3.20) converges for $\tau < \tau_0$ because of the asymptotic properties of $\psi_n(\tau)$ and $\psi_n'(\tau)$.

The lowest value of k for which $\psi_k(\tau)$ occurs is $k = \frac{1}{\mu}$. Consequently as $\tau \to 0$, $z = O(\tau^{\frac{1}{2\mu} - \frac{1}{2}})$. But $\frac{1}{\mu} > 2$ and so $z = 0$ at $\tau = 0$. The origin of coordinates is therefore located at the stagnation point. The distribution of velocity along the axis, that is for $x < 0$, $y = 0$ is obtained by putting $\theta = 0$ in (3.20). This gives

$$z(\tau) = x(\tau) = -E(\tau) \sum_{n=1}^{\infty} \frac{n\mu}{n^2 - \mu^2} e^{-\frac{n s_0}{\mu}} G_{\frac{n}{\mu}}(\tau) \qquad (3.21)$$

where $G_k(\tau) = \psi_k(\tau) + 2\tau\psi_k'(\tau)$.

For large positive k, $G_k(\tau) \sim 2k\tau V(\tau) \frac{ds}{d\tau} e^{ks} > 0$. Consequently as $\tau \to \tau_0$, $x(\tau) \to -\infty$ as required.

The variation of velocity on the side of the wedge is obtained by putting $\theta = \mu\pi$ in (3.20) to give

$$z = e^{i\mu\pi} E(\tau) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n\mu}{n^2 - \mu^2} e^{-\frac{n s_0}{\mu}} G_{\frac{n}{\mu}}(\tau). \qquad (3.22)$$

(3.22) contains the same terms as (3.21) but they now alternate in sign. The series (3.22) oscillates finitely at $\tau = \tau_0$ and gives a finite value of $r = |z|$ at $\tau = \tau_0$ where r denotes distance up the wedge face.

The lines of constant velocity (and consequently of constant pressure), are obtained by putting $\tau = \tau_1$ say and eliminating θ from the two expressions obtained by equating the real and imaginary parts in (3.20).

The expression $\frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$ gives the gradient of the line of constant τ_1 at the point θ. At $\theta = 0, \mu\pi$ this has the values ∞, $\tan\left(\frac{\pi}{2} + \mu\pi\right)$ respectively. That is, the lines of constant velocity are perpendicular to the straight streamlines.

That this must be so is clear on general grounds. For let Φ, Ψ be the velocity potential and stream function of a flow which includes a streamline which is straight for part of its length. Take axes at a point O on its straight part such that Ox is along the streamline.

Now $\left(\frac{\partial\Phi}{\partial y}\right)_x = \left(\frac{\partial\Phi}{\partial\tau}\right)_\theta\left(\frac{\partial\tau}{\partial y}\right)_x + \left(\frac{\partial\Phi}{\partial\theta}\right)_\tau\left(\frac{\partial\theta}{\partial y}\right)_x$.

At O, $\frac{\partial\Phi}{\partial y} = 0$ by the choice of axes; also $\frac{\partial\Psi}{\partial\tau}$ and consequently $\frac{\partial\Phi}{\partial\theta} = 0$. $\frac{\partial\Phi}{\partial\tau}$ does not vanish in general. Therefore $\frac{\partial\tau}{\partial y} = 0$ which is equivalent to the statement made above.

In the solution appropriate to $\tau > \tau_0$ it is clear that the streamline $\Psi = 0$ is continued along $\theta = \mu\pi$ for $\tau > \tau_0$. Care must be taken in the interpretation of the solution, however, especially when the supersonic region is reached, with the consequent danger of limit lines occurring. The significance of singularities is most easily seen from the following relation (3.23).

$$\text{Since} \quad d\Psi = \Psi_r\, d\tau + \Psi_\theta\, d\theta$$
$$d\phi = \phi_\tau\, d\tau + \phi_\theta\, d\theta ,$$
$$\Psi_\theta\, d\phi - \phi_\theta\, d\Psi = \frac{\partial(\phi, \Psi)}{\partial(\tau, \theta)}\, d\tau .$$

Now along a streamline, $d\Psi = 0$ and $d\phi = q_m \tau^{\frac{1}{2}} dS$, dS being the element of length along the streamline.

That is

$$dS = \frac{\rho_0}{2\rho q_m \tau^{\frac{3}{2}}} \, \frac{(M^2-1)\Psi_\theta^2 - 4\tau^2 \Psi_\tau^2}{\Psi_\theta}\, d\tau . \tag{3.23}$$

On a straight streamline $\Psi_\tau = 0$. That is

$$dS = \frac{\rho_0}{\rho q_m} \, \frac{(M^2-1)\, \Psi_\theta}{2\tau^{\frac{3}{2}}}\, d\tau . \tag{3.24}$$

It follows that if $M = 1$ or $\Psi_\theta = 0$ on the wedge face then $\frac{dS}{d\tau} = 0$. In either case let this happen at $r = r_1$, $\tau = \tau_1$. The velocity has reached a maximum on the streamline at that point. Since r is a single-valued function of τ it is impossible to continue the streamline for $r > r_1$ with $\tau < \tau_1$. If values of $\tau > \tau_1$ are inserted in the position coordinates, values of $r < r_1$ are obtained and the streamline apparently turns back on itself which is physically unacceptable. There are two possibilities. Either the flow has reached a limit line through which it cannot be continued; or the streamline ceases to be straight after the point $r = r_1$. Both these cases can occur and will be discussed more fully later. The point to be made at present is that although there exist values of τ which cause Ψ to vanish on $\theta = \mu\pi$ it must not be presumed from this that the streamline $\Psi = 0$ in the physical plane does necessarily take these values.

Near the stagnation point,

$$\Psi \approx c^2 e^{-\frac{S_0}{\mu}} \psi_{\frac{1}{\mu}}(\tau) \sin\frac{\theta}{\mu}$$

$$\approx c^2 e^{-\frac{S_0}{\mu}} \tau^{\frac{1}{2\mu}} \sin\frac{\theta}{\mu}$$

$$\therefore \left[\Psi_\theta\right]_{\theta=\mu\pi} = -O\left(\tau^{\frac{1}{2\mu}}\right).$$

Hence on the wedge face, near the stagnation point, $\frac{dr}{d\tau} = +O\left(\tau^{\frac{1}{2}\left(\frac{1}{\mu}-3\right)}\right)$ using (3.24) or by direct differentiation of (3.22).

The acceleration is given by $q\frac{dq}{dr} = \frac{1}{2}q_m^2\frac{d\tau}{dr} = +O\left(\tau^{\frac{1}{2}\left(3-\frac{1}{\mu}\right)}\right)$. At the wedge tip it is therefore infinite for $\mu < \frac{1}{3}$, that is for wedges of semi-angle $< 60^\circ$ and is zero for wedges of semi-angle $> 60^\circ$.

The result may be interpreted in terms of pressure. Bernouilli's equation gives

$$q\,dq + \frac{dp}{\rho} = 0$$

$$\therefore \frac{1}{\rho}\frac{dp}{dr} = -q\frac{dq}{dr} = -\frac{1}{2}q_m^2\frac{d\tau}{dr}$$

$$\therefore \frac{dp}{dr} = -O\left(\tau^{\frac{1}{2}\left(3-\frac{1}{\mu}\right)}\right)$$

and $$\frac{d^2p}{dr^2} = \operatorname{sgn}\left(\frac{1}{\mu}-3\right)\, O\left(\tau^{2-\frac{1}{\mu}}\right).$$

If $\Psi_\theta \neq 0$ at some intervening point, the pressure gradient is negative infinite both at the stagnation point and at $M = 1$. The pressure curve must therefore have an inflexion in between. This is verified experimentally by the interferometer experiments described by Pack (16) which also show steep pressure gradients near the sonic point. For wedges of semi-angle between 60° and 90°, the pressure gradient is zero at the stagnation point and the second derivative is negative infinite. Few experiments appear to have been done on such large angle wedges but an interesting diagram of the pressure distribution on a wedge of semi-angle 90° occurs in Fig. 14 of a report by W.C. Griffith (26) which lends some support to the above prediction.

Continuation for μ Rational. It is convenient at this stage to find the continuation of the stream function for rational values of μ. This may be obtained by evaluating the residues at the double poles or by regarding $\frac{1}{\mu}$ as the limit of a non-rational sequence of numbers and proceeding to the limit. The second method is chosen.

Accordingly let $\frac{1}{\mu} = \frac{k}{l} + x$ where k and l are integers and let $x \to 0$ through non-rational values.

The terms of the first series of (3.18) for which n is not a multiple of l and those of the second series for which m is not a multiple of k are non-singular; for the rest, combining the $(rl)^{th}$ term of the first series with the $(rk-1)^{st}$ term of the second series it is necessary to consider as a typical expression

$$\lim_{x \to 0} \left[\psi_{-r(k+xl)}(\tau)\, e^{(s_0+i\theta)r(k+lx)} + \frac{\pi l}{k+lx} \, \frac{rk\, C_{rk}\, \psi_{rk}(\tau)\, e^{rk(s_0+i\theta) - \frac{rkl\pi i}{k+lx}}}{\sin \frac{rkl\pi}{k+lx}} \right]$$

$\psi_{-r(k+xl)}(\tau)$ can be written as $\dfrac{k\, C_{rk}\, \psi_{rk}(\tau)}{lx} + \psi_{rk}^{*}(\tau) + rk\, C_{rk} \dfrac{d}{d(rk)} \psi_{rk}(\tau) + C_{rk}\, \psi_{rk}(\tau) + O(x)$. This is apparent from the properties of $\psi_n^{*}(\tau)$ as described in Chapter II.

The principal part at $x = 0$ disappears and the limit becomes, on simplification,

$$e^{rk(s_0+i\theta)} \psi_{rk}^{*}(\tau) + C_{rk} \frac{d}{d(rk)} \left\{ rk\, \psi_{rk}(\tau)\, e^{rk(s_0+i\theta)} \right\} - i r l \pi\, C_{rk}\, e^{rk(s_0+i\theta)} \psi_{rk}(\tau)$$

By summing such terms, W_2 is obtained and the stream function from it by extracting the imaginary part.

Finally

$$\begin{aligned} \Psi_2 = c^2 \Bigg[& \frac{\pi l}{k} \sum_{n=1}^{\infty}{}' \frac{n C_n \psi_n(\tau)\, e^{n s_0} \sin n\left(\theta - \frac{\pi l}{k}\right)}{\sin \frac{n \pi l}{k}} + \sum_{n=1}^{\infty}{}'' \psi_{-\frac{nk}{l}}(\tau)\, e^{\frac{nk s_0}{l}} \sin \frac{nk\theta}{l} \\ & + \sum_{n=1}^{\infty} \psi_{nk}^{*}(\tau)\, e^{nks_0} \sin nk\theta + \sum_{n=1}^{\infty} C_{nk} \frac{d}{d(nk)} \left\{ nk\, \psi_{nk}(\tau)\, e^{nks_0} \sin nk\theta \right\} \\ & - l\pi \sum_{n=1}^{\infty} n\, C_{nk}\, e^{nks_0} \psi_{nk}(\tau) \cos nk\theta \Bigg], \end{aligned} \qquad (3.25)$$

where Σ' denotes summation with terms for which n is a multiple of k omitted and Σ'' similarly for terms for which n is a multiple of l.

The absolute convergence of the series is established for all values of τ such that $\tau_0 < \tau < 1$ and for all θ by means of the asymptotic forms of the functions appearing in it. The continuation is therefore complete.

The last two terms of (3.25) may be written as $\sum_{n=1}^{\infty} C_{nk} \frac{d}{d(nk)} \{ nk\,\psi_{nk}(\tau)\, e^{nks_0} \} \sin nk\theta + \sum_{n=1}^{\infty} C_{nk}\, nk\, \psi_{nk}(\tau) \times e^{nks_0} (\theta - \frac{l\pi}{k}) \cos nk\theta$. This shows that $\Psi_2 = 0$ for $\theta = \frac{l\pi}{k}$ as is required for the streamline to continue up the side of the wedge.

<u>Boundedness of the Supersonic Region</u>. Despite the complexity of the series (3.25) it is the more convenient one to use to show that the supersonic region is bounded. The reason for this is that convergence properties for series such as (3.19) and its derivatives are difficult to establish because the functions are taken at points arbitrarily near singularities. However, if the series for z corresponding to (3.25) is taken and the asymptotic forms used, it follows that for any θ, $|z|$ is bounded for any subsonic or any supersonic value of τ (provided $\tau < 1$). Asymptotic formulae for $\tau = \tau_s$ are not so readily available

but the boundedness of $|z|$ for subsonic and supersonic τ gives a bounded value of $|z|$ for the sonic line by continuity.

The solution therefore consists of a supersonic region embedded in a subsonic region. It is true, just as in Ringleb's solution, that there will be values of x, y in the subsonic region which are also associated with supersonic values of τ. This is due to τ, θ being many-valued functions of x, y. The whole plane (outside the wedge) can be covered once by a solution which has $\tau < \tau_s$ everywhere to the outside of the sonic line and this is the one discussed. The velocity far away from the wedge tends to τ_0, since $|z| \to \infty$ when $\tau \to \tau_0, \theta \to 0$.

Although a limit line occurs in the supersonic region, there will be a limiting streamline beyond which the flow is non-singular. It is to be expected that this limiting streamline attains supersonic velocity at some point of its path.

Whether a limit line appears in the solution or not, it is evident that there is now a discrepancy between the shape of $\Psi = 0$ in the original and modified problem. In the original incompressible problem the streamline turned the corner with infinite velocity and then became straight and parallel to the axis of the wedge again. But if $\theta = 0$

is substituted in (3.19) the result is

$$\Psi_2 = -c^2\mu\pi \sum_{n=2}^{\infty} n C_n \psi_n(\tau) e^{ns_0}$$

In (3.25) it is

$$\Psi_2 = -\frac{c^2\pi l}{k} \sum_{n=2}^{\infty}{}' n C_n \psi_n(\tau) e^{ns_0} - c^2 l\pi \sum_{n=1}^{\infty} n C_{nk} e^{nks_0} \psi_{nk}(\tau)$$

$$= -\frac{c^2 l\pi}{k} \sum_{n=2}^{\infty} n C_n \psi_n(\tau) e^{ns_0}$$

which is the same expression.

This is certainly not zero as τ varies so that the streamline $\Psi = 0$ does not continue parallel to the axis in the right hand part of the physical plane. The further interpretation of the solution is postponed until the significance of the singularities has been more fully investigated.

CHAPTER IV.

DISCUSSION OF THE SINGULARITIES.

General Theory. The hodograph transformation is a method of obtaining a solution of a flow problem by representing the stream function Ψ as a function of the velocity coordinates τ and θ . The streamlines are represented by curves in the τ , θ or alternatively in the u, v plane. To find the streamlines in terms of the position coordinates it is necessary to obtain the mapping of these curves in the x, y plane. The required transformation is given by (1.28).

The mapping will not in general be everywhere one-one. And since it is necessary for physical continuity that u and v should be single-valued functions of x and y, some care is required in considering which branch of the function is mapped. It is therefore essential to discover where the singularities of the transformation exist and what effect they have on the flow pattern in the x, y plane.

All possible singularities of the hodograph transformation have been examined by Craggs (5). A summary of the general theory is now given with particular reference to the types of singularity which occur in the present problem.

In a general transformation from the u, v to the x, y plane, singularities will occur where the Jacobian $J = \frac{\partial(x,y)}{\partial(u,v)}$ becomes zero or infinite.

J can be written as $\frac{\partial(x,y)}{\partial(\phi,\Psi)} \frac{\partial(\phi,\Psi)}{\partial(\tau,\theta)} \frac{\partial(\tau,\theta)}{\partial(u,v)} = \left(\frac{\rho_0}{\rho q_m^2 \tau}\right)^2 \left\{(M^2-1)\,\Psi_\theta^2 - 4\tau^2\,\Psi_\tau^2\right\}$. This becomes infinite in general when $\rho = 0$ or $\tau = 0$ or when Ψ_θ or Ψ_τ is infinite.

The case $\rho = 0$ corresponds to $\tau = 1$ or $q = q_m$. This is not of interest in the present problem where $\tau \neq 1$ in a finite part of the plane. $\tau = 0$ only at an isolated point and the flow there is similar to the flow at a stagnation point in incompressible flow.

If Ψ_τ and Ψ_θ do not both vanish, J cannot vanish unless $M > 1$. In this case J vanishes along a curve and gives rise to a limit line in the supersonic part of the flow field. Ψ_τ and Ψ_θ can both vanish only at an isolated point.

Suppose $J = 0$ along a curve C in the u, v plane. Let C' be its image in the x, y plane. Then there is the following general result.

All curves which meet C at a point P say in a certain privileged direction have images in the x, y plane which are cusped at P', the image of P. (It is assumed $\frac{\partial J}{\partial u}$, $\frac{\partial J}{\partial v}$ do not both vanish at P). The images of all other curves through P are not cusped but touch C' at P'.

A neighbourhood of P corresponds to a neighbourhood of P' confined to one side of C' and doubly covered so that the x, y plane may be thought of as being folded over along C'.

If the point P is such that the privileged direction coincides with the direction of C at P, then C' is cusped at P' and the neighbourhood of P' is triply covered.

It can be shown that C' is always tangential to one of the characteristic directions at P', say that of the first family. The characteristics in the x, y plane have different directions at P' and the angle between them is bisected by the streamline at P'. It follows from the general result that the streamline at P' and the characteristic of the second family are both cusped since neither is tangential to C' at P'.

The characteristics in the u, v plane are independent of the solution considered. They consist of two families of similar epicycloids which start from the sonic circle at right angles to it and finally touch the circle $q = q_m$. The lines of constant τ and θ are also fixed. Since the characteristic of the second family in the x, y plane is cusped at P', it follows that its image (which is one of the fixed epicycloids) determines the privileged direction in the u, v plane. The streamline in the u, v plane must also be in the privileged direction and must therefore touch the characteristic of the second family at P. This might be used to detect the presence of singularities by examining conditions in the hodograph plane alone.

In general, the directions of the lines of constant τ and of constant θ in the u, v plane nowhere coincide with the characteristic directions. (This is true everywhere if $\tau \neq \tau_s$ or 1). The lines of constant τ and θ are therefore not in the privileged direction and it follows that their images in the x, y plane envelop C'.

When J = ∞ (corresponding to Ψ_θ or Ψ_τ becoming infinite) the situation is not quite the same as J = 0 with x, y and u, v interchanged. For the characteristics in the u, v plane, being fixed, can never have an envelope (except for $q = q_m$) and the singular line therefore consists of a definite characteristic in the u, v plane; and consequently its image is a characteristic in the x, y plane. The privileged direction at a point is that of the other characteristic at the point. Streamlines in the u, v plane touch the singular characteristic and lines of constant τ and θ are cusped there, although in this case the two branches of the cusp are identical. It follows that in the x, y plane the lines of constant τ and θ at a point on the singular line touch the second characteristic through the point.

Since Ψ_θ and Ψ_τ are finite at all finite points of the plane in the wedge problem this will not be discussed further. (This fact may be seen from the series for Ψ when $\tau \neq \tau_0$ and from the integral form for Ψ at $\tau = \tau_0$). Singularities corresponding to J infinite appear to be absent in most problems of flow past a body but they do occur in transonic flow through a nozzle. This case has been treated by Lighthill (12).

<u>Application to the Wedge Problem</u>. The types of singularity of interest which might occur in the present problem are therefore

(i) isolated singularities where $\Psi_\tau = \Psi_\theta = 0$,

(ii) a limit line in the supersonic region in the physical plane corresponding to the image of the curve in the u, v plane given by $(M^2-1)\ \Psi_\theta{}^2 - 4\tau^2\ \Psi_\tau{}^2 = 0$.

On a straight streamline $\Psi_\tau \neq 0$. If, when $M < 1$, there is a point on the wedge face where $\Psi_\theta = 0$, an isolated singularity will occur. It has already been pointed out in Chapter III that the streamline cannot continue straight after this point. However, since the singularity is isolated, a streamline arbitrarily near $\Psi = 0$

encounters no singularity and therefore $\Psi = 0$ is a limit of non-singular streamlines and must be capable of some physically reasonable interpretation. It will be shown that $\Psi = 0$ is continued as a curve and that the slope of the streamline is continuous.

Alternatively if $\Psi_\theta \neq 0$ on the wedge side for $M < 1$, the first singularity encountered will be at the sonic point which is part of a limit line. It will now be established that both these cases can occur.

First of all it is shown that Ψ_θ cannot vanish on the side of the wedge before the point $\tau = \tau_0$ is reached. To prove this, it must be shown that $[\Psi_\theta]_{\theta = \mu\pi} < 0$ for all τ such that $0 < \tau < \tau_0$.

$$\Psi = c^2 \sum_{n=1}^{\infty} \psi_{\frac{n}{\mu}}(\tau)\, e^{-\frac{n s_0}{\mu}} \sin\frac{n\theta}{\mu} \qquad (0 < \tau < \tau_0)$$

$$\therefore [\Psi_\theta]_{\theta=\mu\pi} = c^2 \sum_{n=1}^{\infty} \frac{n}{\mu}(-1)^n \psi_{\frac{n}{\mu}}(\tau)\, e^{-\frac{n s_0}{\mu}}$$

$$= c^2 \sum_{n=1}^{\infty} \frac{n}{\mu}(-1)^n e^{\frac{n}{\mu}(s-s_0)} \left\{ 1 + n \sum_{m=2}^{\infty} \frac{C_m e^{ms} \psi_m(\tau)}{n + m\mu} \right\} \text{ by } (2.11)$$

$$= \frac{c^2}{\mu} \sum_{m=2}^{\infty} R_m \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n+\alpha_m} x^n + \frac{c^2}{\mu} \sum_{n=1}^{\infty} (-1)^n n x^n$$

where $R_m = C_m e^{ms} \psi_m(\tau)$ which is positive (see (13)), $\alpha_m = m\mu$, $x = e^{\frac{s-s_0}{\mu}} < 1$. (The order of summation may be inverted because of the absolute convergence of the series involved).

Now $\sum_{n=1}^{\infty} (-1)^n n x^n = -\frac{x}{(1+x)^2} < 0$, so that the expression will certainly be negative if $\sum_{n=1}^{\infty}(-1)^n \frac{n^2x^n}{n+\alpha} < 0$ for all $\alpha > 0$ and all x in $0 < x < 1$.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2x^n}{n+\alpha} = \sum_{n=1}^{\infty}(-1)^n\left\{nx^n - \alpha x^n + \frac{\alpha^2x^n}{n+\alpha}\right\}$$

$$= -\frac{x}{(1+x)^2} + \frac{\alpha x}{1+x} + \frac{\alpha^2}{x^\alpha}\int_0^x \frac{t^{\alpha-1}}{1+t}\,dt - \alpha$$

$$= -\frac{x}{(1+x)^2} - \frac{\alpha}{1+x} + \frac{\alpha}{1+x}\left[1 + \frac{1!}{\alpha+1}\frac{x}{1+x} + \frac{2!}{(\alpha+1)(\alpha+2)}\left(\frac{x}{1+x}\right)^2 + \cdots\right]$$

on repeated integration by parts. It is therefore to be shown that

$$\frac{x}{(1+x)^2} > \frac{\alpha}{\alpha+1}\frac{x}{(1+x)^2} + \frac{\alpha\, 2!}{(\alpha+1)(\alpha+2)}\frac{x^2}{(1+x)^3} + \frac{\alpha\, 3!}{(\alpha+1)(\alpha+2)(\alpha+3)}\frac{x^3}{(1+x)^4} + \cdots$$

i.e. that

$$\frac{1}{\alpha} > \frac{2!}{\alpha+2}\left(\frac{x}{1+x}\right) + \frac{3!}{(\alpha+2)(\alpha+3)}\left(\frac{x}{1+x}\right)^2 + \frac{4!}{(\alpha+2)(\alpha+3)(\alpha+4)}\left(\frac{x}{1+x}\right)^3 + \cdots = S(x,\alpha).$$

This will certainly be true for all x in $0 < x < 1$ if $\frac{1}{\alpha} > S(1,\alpha)$. Now

$$\frac{1}{\alpha} - S(1,\alpha) = \frac{1}{\alpha} - \frac{2!}{\alpha+2}\cdot\frac{1}{2} - \frac{3!}{(\alpha+2)(\alpha+3)}\cdot\frac{1}{2^2} - \cdots$$

$$> \frac{1}{\alpha} - \frac{2!}{\alpha+2}\cdot\frac{1}{2} - \frac{3!}{(\alpha+2)(\alpha+3)}\frac{1}{2^2} - \frac{4!}{(\alpha+2)(\alpha+3)(\alpha+4)}\left(\frac{1}{2^3}+\frac{1}{2^4}+\cdots\right)$$

$$= \frac{(\alpha+12)(\alpha+4)-12\alpha}{2\alpha(\alpha+2)(\alpha+3)(\alpha+4)} > 0,$$

which gives the required result.

It follows that $\Psi_\theta < -\frac{c^2x}{\mu(1+x)^2} < 0$ on the wedge side.

Moreover when $r = r_0$,

$$\Psi_\theta < -\frac{c^2}{4\mu} \quad \text{and this is independent of } r_0 .$$

Now by taking τ_0 close enough to τ_s it follows by continuity that $[\Psi_\theta]_{\theta=\mu\pi} < 0$ for all τ such that $0 < \tau \leq \tau_s$. In other words, it is possible, by taking τ_0 large enough, to ensure that the first singularity on the wedge face occurs at the sonic point.

To show that an isolated singularity can occur before the sonic point is reached, it is necessary to prove that for some τ_0 and μ, $[\Psi_\theta]_{\theta=\mu\pi}$ is positive for some τ such that $\tau_0 < \tau < \tau_s$.

Using the form (3.25) with $\frac{1}{\mu} = \frac{k}{l}$,

$$[\Psi_\theta]_{\theta=\mu\pi} = c^2\left[\frac{\pi l}{k}\sum_{n=1}^{\infty}{}' \frac{n^2 C_n \psi_n(\tau) e^{ns_0}}{\sin\frac{n\pi l}{k}} + \sum_{n=1}^{\infty}{}'' \frac{nk}{l}\psi_{-\frac{nk}{l}}(\tau)\, e^{\frac{nks_0}{l}}(-1)^n \right.$$
$$\left. + \sum_{n=1}^{\infty}(-1)^{nl}\, nk\, \psi_{nk}^{*}(\tau)\, e^{nks_0} + \sum_{n=1}^{\infty} C_{nk}\frac{d}{d(nk)}\left\{n^2k^2(-1)^{nl}\psi_{nk}(\tau)\, e^{nks_0}\right\}\right] \qquad (4.1).$$

All the series concerned are absolutely convergent. Consider k, l and τ fixed and rearrange the terms as a generalized power series in e^{s_0}. By taking s_0 small enough, the sign of (4.1) will be the same as the sign of the first term of the new series. But this term is clearly $\frac{c^2\pi l}{k}\,\frac{4C_2\psi_2(\tau)e^{2s_0}}{\sin\frac{2\pi l}{k}}$ since $\frac{k}{l} > 2$; and this is positive. Hence $[\Psi_\theta]_{\theta=\mu\pi}$ must vanish somewhere between τ_0 and τ_s if τ_0 is small enough.

The same result can be obtained by keeping s_0 and τ fixed and letting $\frac{k}{l}$ become large, that is letting the wedge angle become small.

To do this, keep l fixed and let k become large through a sequence of integers prime to l.

Using the asymptotic forms for the terms of the series in (4.1), it is clear that the sums of all the series except the first become arbitrarily small as $k \to \infty$. Hence, given $\varepsilon > 0$, there exists a positive integer K such that their combined sum $< \frac{\varepsilon}{2}$ when $k > K$.

Now $\frac{1}{k} \Big/ \sin\frac{n\pi l}{k}$ is bounded for all k when l, n are integers, l is prime to k and n is not a multiple of k.

Hence N exists independent of k such that

$$\left| \sum_{n=r}^{\infty} \frac{c^2\pi l}{k} \frac{n^2 C_n \psi_n(\tau) e^{n s_0}}{\sin(n\pi l/k)} \right| < \frac{\varepsilon}{2} \quad \text{whenever } r \geq N.$$

Now let k be $>$ Max. (K, Nl). Then the first N terms of the first series of (4.1) are all positive and all the remaining terms, together with the other series, have modulus $< \varepsilon$.

The first term is not arbitrarily small since it tends to $2c^2 C_2 \psi_2(\tau) e^{2s_0}$ as $k \to \infty$.

Therefore (4.1) is positive for large enough k.

To recapitulate, the following has been established.

If the infinity velocity τ_0 is high enough, the first singularity which appears on the wedge side is at the sonic point. However, by taking either the infinity velocity or the wedge angle sufficiently small, an isolated singularity can be made to occur on the wedge side at a velocity below any prescribed velocity, in particular below the velocity of sound. In no case, however, does this singularity occur before the velocity has attained on the side of the wedge its velocity at infinity upstream.

Method of Investigating the Isolated Singularity. The interpretation of the solution when Ψ_θ vanishes on the wedge side before the sonic point is reached will be examined first. As this occurs when $\tau > \tau_0$ the appropriate series to use is Ψ_2. It is a serious drawback that in either of its forms (that is for μ rational or non-rational) the expression is highly complicated. (3.19) is simpler in form but is unsuitable to use numerically because of the uncertain behaviour of the terms near their singularities. Even if it were possible to approximate by taking a suitable number of terms from each of the series in (3.19), (in a manner indicated by (3.17)), the terms would have to be computed individually and convergence is certainly slow near $\tau = \tau_0$. The same is true of (3.25) where the elaborate nature of the series involved makes computing practically impossible. It is possible to use the integral which gave the continuation for $\tau_0 < \tau < \tau_s$ and $\theta \neq 0$ but here too the difficulties are formidable.

Nevertheless it is possible to reproduce the principal features of a singularity of this type by taking a model which behaves similarly to the given flow in the neighbourhood of the singularity. This is set up in the following way.

It is easily verified that a solution of the equation for the stream function of the form $\Psi = \psi_n(\tau) \sin n\theta$ where $n > 1$, represents flow in a corner of angle $\frac{\pi}{n}$ (see for example (10)). The flow has limit lines but in the neighbourhood of the corner it gives the appropriate solution. When $n < 0$, and is not an integer < -1, the solution represents approximately flow round a reflex angle of magnitude $\pi - \frac{\pi}{n}$. Here the velocity is zero at infinity and the streamline $\Psi = 0$ cannot be continued right up to the corner because a limit line occurs. However, if the first non-singular streamline is taken as the fixed boundary, the solution represents flow round a curved profile with a total deflection of $-\frac{\pi}{n}$. A classic example is Ringleb's solution which consists of a combination of $\psi_{-1}(\tau)$ and $\psi_1(\tau)$ multiplied by $\sin\theta$. The case $n = -\frac{3}{5}$ mentioned in Chapter II affords another example. This solution can be readily evaluated because of the simplicity of $\psi_{-\frac{3}{5}}(\tau)$ for $\gamma = 1.4$.

In these cases the streamline $\Psi = 0$ encounters a limit line because Ψ_θ does not vanish on it for any

$\tau < \tau_s$. The object of the present work is to construct an example in which Ψ_θ vanishes on the straight streamline $\Psi = 0$ (i.e. $\Psi_\theta = \Psi_\tau = 0$) for some $\tau < \tau_s$. The conditions are then similar to those which arise in the wedge problem and the evaluation of this very much simpler solution will serve as a guide to interpret the configuration which arises in a corresponding case in the wedge problem.

<u>The Simplified Problem.</u> The particular example evaluated is

$$\Psi = -\psi_{-13.5}(\tau)\sin 13.5\theta + 10^6\psi_{-4.5}(\tau)\sin 4.5\theta . \quad (4.2).$$

Since $\psi_{-n}(\tau) \sim \tau^{-\frac{n}{2}}$ as $\tau \to 0$, Ψ is dominated by the first term near $\tau = 0$. Since small values of τ correspond to large values of $|z|$, the second term in (4.2) may be regarded as a perturbation on the solution which consists of the first term by itself. The expression (4.2) was so chosen because the functions $\psi_{-4.5}(\tau)$ and $\psi_{-13.5}(\tau)$ and their derivatives are readily obtainable from the Huckel Tables (17).

The points on the streamline $\Psi = 0$ for which $\theta = 0$ are now obtained from (1.28). On $\theta = 0$, $y = 0$ and

$$x(\Upsilon) = E(\Upsilon)\Big[\Big\{\frac{6.45\,\psi_{-13.5}(\Upsilon) + \Upsilon\,\psi'_{-13.5}(\Upsilon)}{12.5} - \frac{6.45\,\psi_{-13.5}(\Upsilon) - \Upsilon\,\psi'_{-13.5}(\Upsilon)}{14.5}\Big\}$$

$$-10^6\Big\{\frac{2.25\,\psi_{-4.5}(\Upsilon) + \Upsilon\,\psi'_{-4.5}(\Upsilon)}{3.5} - \frac{2.25\,\psi_{-4.5}(\Upsilon) - \Upsilon\,\psi'_{-4.5}(\Upsilon)}{5.5}\Big\}\Big]. \quad (4.3)$$

The numerical details are given in Chapter V but the solution may be discussed qualitatively here.

From (3.24) the relation $\frac{dx}{d\Upsilon} = \frac{\rho_0}{\rho q_-}\frac{(M^2-1)[\Psi_\theta]_{\theta=0}}{2\Upsilon^{\frac{1}{2}}}$ is easily obtained. Now

$$[\Psi_\theta]_{\theta=0} = -13.5\,\psi_{-13.5}(\Upsilon) + 4.5 . 10^6\,\psi_{-4.5}(\Upsilon). \quad (4.4)$$

For small values of Υ this expression is negative. It follows that initially x increases from $-\infty$ as Υ increases from 0.

However at $\Upsilon = \Upsilon_1$, $[\Psi_\theta]_{\theta=0}$ changes sign ; Υ_1 lies between .07 and .08. Thereafter as Υ increases, x decreases and the streamline apparently turns back on itself.

As has already been remarked this cannot be interpreted as a breakdown of the flow such as occurs with a limit line for $\Psi_\Upsilon \neq 0$ off the straight streamline and streamlines arbitrarily close to $\Psi = 0$ encounter no singularity. The explanation is that $\Upsilon = \Upsilon_1$, $\theta = 0$ is a branch point singularity and the apparently absurd values of x just obtained belong to a different branch of the solution. Moreover it must be possible to continue

$\Psi = 0$ in a physically acceptable manner. Obviously it cannot be continued with θ still equal to 0.

Now $\Psi = 0$ when

$$\sin 4.5\theta \left\{ 10^6 \psi_{-4.5}(\tau) - (3 - 4\sin^2 4.5\theta)\psi_{-13.5}(\tau) \right\} = 0$$

i.e. $\sin 4.5\theta = 0$ or

$$\sin^2 4.5\theta = \frac{3\psi_{-13.5}(\tau) - 10^6 \psi_{-4.5}(\tau)}{4\psi_{-13.5}(\tau)} \quad (4.5)$$

A non-zero value of θ on $\Psi = 0$ can be obtained from (4.5) if $\frac{3\psi_{-13.5}(\tau) - 10^6 \psi_{-4.5}(\tau)}{4\psi_{-13.5}(\tau)} > 0$ but < 1.

From the numerical details in the neighbourhood of $\tau = \tau_1$, the first condition is seen to imply the second; and both conditions will be satisfied if $\tau < \tau_1$. Moreover at $\tau = \tau_1$ the solution of (4.5) is $\theta = 0$ but as τ decreases non-zero values of θ satisfy (4.5). The streamline $\Psi = 0$ can therefore be continued as a curve with continuous values of τ and θ. Physical requirements demand this, for the flow cannot turn a sharp corner at a subsonic velocity or flow in the interior of a sharp corner without a stagnation point. For this reason, values of such as $4.5\,\theta = \pi$ which would satisfy $\Psi = 0$ were not considered. Furthermore, the values of x and y would be discontinuous if discontinuities were introduced in θ at $\tau = \tau_1$.

It is to be expected that the streamline will tend to infinity as $\gamma \to 0$. From (4.5), $\sin^2 4.5\theta \to \frac{3}{4}$ as $\gamma \to 0$, i.e. $\theta \to \pi/(-13.5)$.

If now the asymptotic form of the ψ function as $\gamma \to 0$ is inserted in the formulae for the position coordinates with $\theta = \pi/(-13.5)$ it is quickly verified that

$$x = f(\gamma) \cos \pi/13.5$$

$$y = -f(\gamma) \sin \pi/13.5$$

where $f(\gamma) \to \infty$ as $\gamma \to 0$.

A number of values were plotted and $\Psi = 0$ has the form of Fig. (xiv).

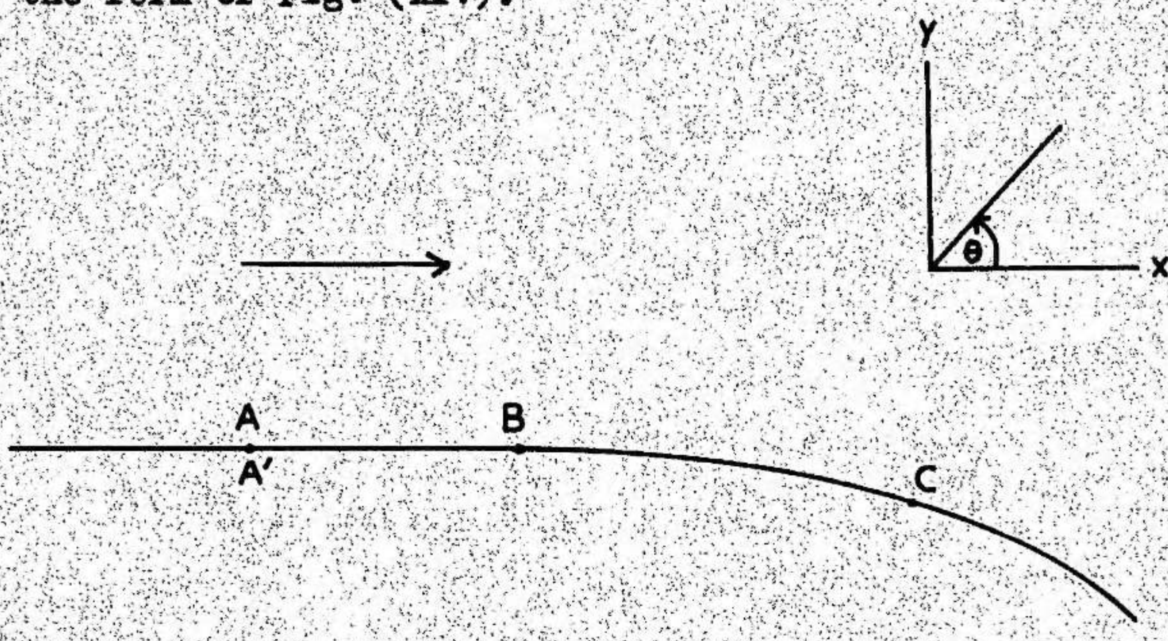


Fig.(xiv)

A negative value of θ was chosen as the solution of (4.5). This was to give continuity as negative values of θ correspond to positive values of y to the left of the line $x = x(\tau_1)$. If positive values of θ were taken throughout, a solution would be obtained which is the reflection of Fig. (xiv) in $y = 0$.

Although it is not of application to the wedge problem, it is of some interest to examine the solution below $\Psi = 0$. Some points were obtained in the numerical work, although this was chiefly to act as a check that all points (τ, θ) which made Ψ negative had images in the physical plane which fell below $\Psi = 0$. But they also serve to reconstruct the solution below ABC in Fig. (xiv). B is the singular point on the streamline $\Psi = 0$, C is a typical point on its curved part and A a typical point on its straight part. A' is the same point as A in the physical plane but is associated with the flow below $\Psi = 0$.

If this flow is considered as being bounded by $\Psi = 0$, the flow round the inside of a bend is obtained. Above ABC, small negative values of θ give Ψ positive when $\tau < \tau_1$. Consequently, to give a negative value of Ψ for these values of θ, τ must be taken $> \tau_1$. It follows that the set of values of x obtained on $\theta = 0$ by putting values of $\tau > \tau_1$ into (4.3) is appropriate to this solution. The two solutions cannot therefore be joined up along $\theta = 0$ although they can be continued over the curved part of $\Psi = 0$. On $\Psi = 0$, as $-x$ becomes large, τ increases but at $\tau = \tau_s$ a limit line is encountered. The flow cannot therefore be continued back along the streamline past the sonic point.

The flow in the hodograph plane is now considered. At the singular point $\Psi_\tau = \Psi_\theta = \Psi_{\tau\tau} = 0$. It is easily verified that at this point $\Psi_{uu}\Psi_{vv} - \Psi_{uv}^2 = -\frac{4\Psi_{\tau\theta}^2}{q^4} < 0$. It is therefore a saddle point in the hodograph plane. Fig. (xv) shows the streamline $\Psi = 0$ in the hodograph plane together with a typical streamline (shown dotted) in the flow above ABC. The points A, A', B, C in Fig. (xv) correspond to their equivalents in Fig. (xiv). O is the origin of coordinates corresponding to $\tau = 0$. The θ scale has been considerably magnified.

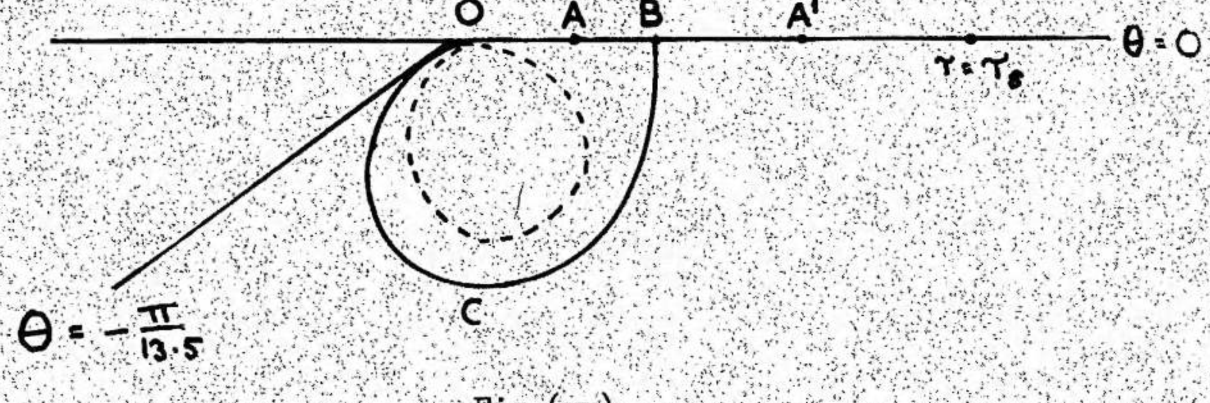


Fig. (xv)

OABCO corresponds to $\Psi = +0$, A'BCO to $\Psi = -0$. As anticipated A, A' are distinct points in the hodograph plane.

This is interesting as an example of the singularity listed in Theorem 4 of Craggs's paper (5). In particular it illustrates how a 360° neighbourhood of the physical plane at B is mapped into a 180° neighbourhood of the hodograph plane. (u, v have to be interchanged with x, y in Craggs's notation to conform to that used here).

Finally, it might be pointed out that the velocity τ_1 is the highest which appears in the upper solution $\Psi \geqslant 0$ and that it occurs only at the point B. For when $\tau = \tau_1$, Ψ may be written as

$$\Psi(\tau_1, \theta) = -\psi_{-13.5}(\tau_1)\left\{\sin 13.5\theta - 3\sin 4.5\theta\right\} = 4\psi_{-13.5}(\tau_1)\sin^3 4.5\theta.$$

Hence for $0 > \theta > -\frac{\pi}{4.5}$, $\Psi(\tau_1, \theta) < 0$. However for the upper solution $\Psi \geqslant 0$ and $0 \geqslant \theta > -\frac{\pi}{13.5}$ (see Fig. (xv)); consequently $\tau \neq \tau_1$ above ABC. However $\tau < \tau_1$ everywhere sufficiently far from the wedge. Hence, by continuity, $\tau < \tau_1$ everywhere above $\Psi = 0$.

Results of the Model Applied to the Wedge Problem. The particular example just described was selected so that the singularity which arose in it represented as closely as possible the conditions at the singularity which was found to occur on the side of the wedge. In either case the flow accelerates along a straight wall until it encounters at a subsonic velocity a point where $\Psi_\theta = \Psi_\tau = 0$. In the simple problem the critical streamline has been shown to continue through the singularity as a curved line which becomes asymptotically straight at infinity and on which the velocity decreases monotonically. It may therefore be conjectured with some confidence that the streamline $\Psi = 0$ in the wedge problem continues as a curved line which tends to become straight ultimately. Moreover it seems plausible that, as in the simple problem, the maximum velocity occurs at the singularity or at least that the flow remains subsonic above the critical streamline. If this is so, the whole flow is non-singular and capable of complete representation by application of (1.28). The solution therefore describes the subsonic flow past a profile consisting of a wedge which degenerates from the shoulder into a curve with a continuous tangent. It is to be expected that the ultimate direction of this curve will be parallel to the direction of the flow at infinity upstream. This can be shown analytically by considering the value of θ which makes $|z|$ infinite at $\tau = \tau_0$. ($|z|$ will not become infinite for any θ when τ lies between τ_0 and 1). The series expansions are not very useful to investigate conditions at $\tau = \tau_0$ but the integral form may be used. The position coordinates are obtained by a straightforward application of (1.28) under the integral sign. Then it is seen that $|z|$ is bounded for any value of θ such that $0 < \theta < 2\mu\pi$. It follows that $\theta = 0$ is the only acceptable value for the asymptotic direction of the streamline. This result is indicated if the solution is considered for small values of τ_0. For then the second series of (3.19) is much smaller numerically than the first, at any rate in the neighbourhood of $\tau = \tau_0$. This is easily seen by considering the asymptotic forms of the individual terms. Hence Ψ_2 is dominated near $\tau = \tau_0$ by the first series alone which gives $\Psi = 0$ for $\theta = 0$. In a similar manner (3.25) is dominated by the series $c^2 \sum_{n=1}^{\infty}{}'' \psi_{-\frac{nk}{l}}(\tau) e^{\frac{nks_0}{l}} \sin \frac{nk\theta}{l} + c^2 \sum_{n=1}^{\infty}{}'' \psi_{nk}(\tau) e^{nks_0} \sin nk\theta$ which also vanishes for $\theta = 0$. It is natural as τ_0 decreases that the incompressible flow pattern should tend to reappear for a decrease in τ_0 may be interpreted not necessarily as a decrease in the <u>actual</u> velocity q_0 but as an increase in q_m, that is a tendency to diminish the effects of compressibility in the fluid. (See footnote on page 14).

The Limit Line Solution. The second alternative is now discussed. That is, it is assumed that $\Psi_\theta \neq 0$ on $\theta = \mu\pi$ for $\tau \leq \tau_s$. In this case the singularity is not confined to an isolated point but $J = 0$ is a curve in the hodograph plane passing through $\tau = \tau_s$, $\theta = \mu\pi$. Its image in the physical plane is a limit line existing in the supersonic region and therefore bounded (for the particular branch of the stream function which is selected and to which the analysis has been confined). As has already been remarked, there will exist a limiting streamline $\Psi = \Psi_0$ which will not meet the limit line and which can therefore be regarded as a fixed boundary above which the flow is free of singularities. All the streamlines lying between $\Psi = 0$ and $\Psi = \Psi_0$ are cusped on meeting the limit line and this is, of course, physically meaningless.

It is natural to ask whether or not it is possible to find a continuation of the flow through the limit line. It is well known that consistent supersonic flow patterns can be constructed by "patching" suitable flows together along a characteristic, requiring only continuity of the velocity components across the characteristic and not necessarily of their derivatives. At first sight this might seem to provide a method of overcoming the difficulty here. For a limit line is known to be everywhere tangential to one of the characteristic directions at each point of it; and it might prove possible to patch a suitable flow onto this limit lihe to join up with the oncoming branch of each (cusped) streamline. However this cannot in fact be done and Tollmien (18) has shown that any continuation through the limit line is impossible.

It is natural to assume that the occurrence of limit lines in the solution is associated with the appearance of shock waves in the actual physical problem. In a sense, however, it is meaningless to talk of the "actual" physical problem. For a given solution in the hodograph variables is reconstructed a posteriori and nothing can therefore be deduced about the occurrence of shock waves in a given physical problem. It is impossible, moreover, to interpret the limit line itself as a shock wave for it is everywhere tangential to a characteristic direction. This means that if a shock wave were to replace the limit line it would be everywhere inclined to the flow direction at the Mach angle and so would be of zero strength.

It is clear therefore, that nothing can be done if the solution is used to represent the flow right up to the limit line. An alternative at once suggests itself. Apart from the point $\tau = \tau_s$, $\theta = \mu\pi$, the limit line lies wholly within the supersonic region. Thus it might be possible to draw a line between the sonic line and the limit line, accept the solution outside this line and attempt to continue the flow inside by using characteristics and possibly introducing shock waves. Furthermore, because of the hyperbolic nature of the differential equation in the supersonic region, some choice is available. It might, for example, be possible to satisfy the condition that the flow turned the corner of the wedge and continued parallel to the main flow, at any rate initially. To achieve this a point-localized Prandtl-Meyer expansion would have to be inserted at the point corresponding to $\tau = \tau_s$, $\theta = \mu\pi$, which would therefore be identified as the corner of the wedge.

It is interesting to note that in such a solution the sonic point would be located at the corner of the wedge. This fact has been supported on theoretical grounds by Busemann (19) and Maccoll (20) and is indicated by the available experimental evidence. Some small discrepancies have been observed in the more accurate experiments, but these are believed to be due to boundary layer effects.

In any such process of modifying the solution by means of characteristics the whole subsonic region must be left unaltered; for the whole subsonic domain is simply connected because of the boundedness of the supersonic region.

Therefore it is certainly not possible to continue the streamline $\Psi = 0$ as a straight line in the subsonic region as it has already been shown that for $\tau > \tau_0$, $\theta = 0$ does not give $\Psi = 0$. The boundary which the expression gives as $\Psi = 0$ must be preserved.

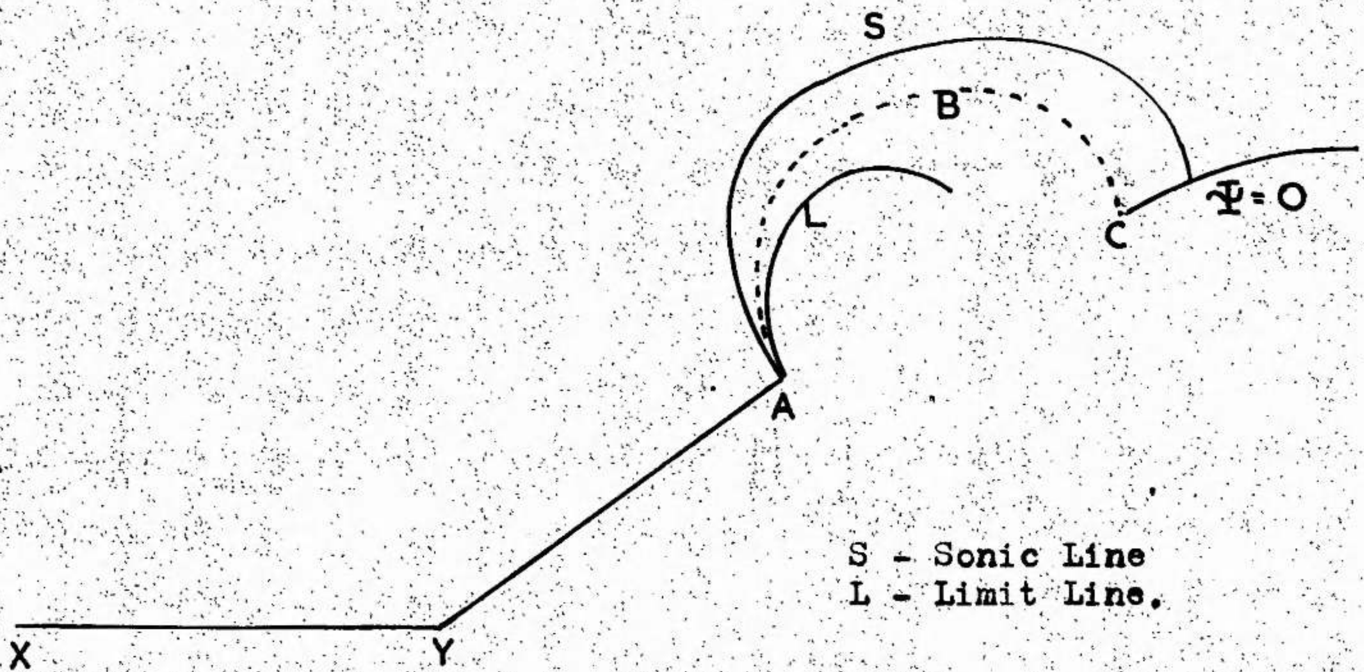


Fig.(xvi)

It might still be possible to select a line ABC lying between the sonic and limit lines, accept the flow outside ABC and continue the flow inside ABC by characteristics (Fig. (xvi)). An attempt might be made to insert a point-localized Prandtl-Meyer expansion at A and to continue the flow parallel to XY immediately after A, thus reproducing accurately conditions at the shoulder of a wedge and representing completely flow past a wedge shaped body with some undulation of the upper surface which starts at a finite distance downstream from the shoulder.

It will now be shown that this cannot be done exactly. In the first place shock waves cannot be introduced into the flow in ABC. For any gas passing through a shock wave suffers an increase in entropy which it retains as it enters the subsonic region downstream and the whole of the subsonic region must be at the same entropy level.

To see more clearly the significance of introducing a point-localized Prandtl-Meyer expansion into the flow at A, it is necessary to consider the configuration in the hodograph plane (Fig. (xvii)).

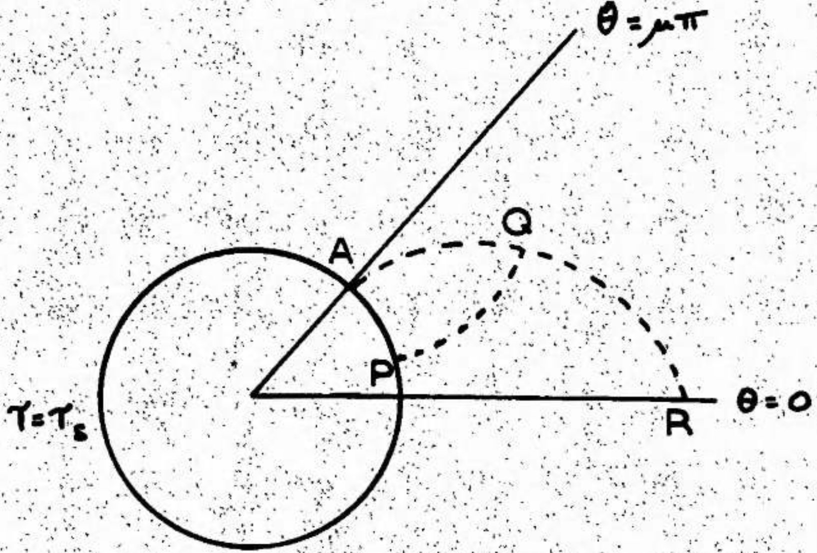


Fig. (xvii)

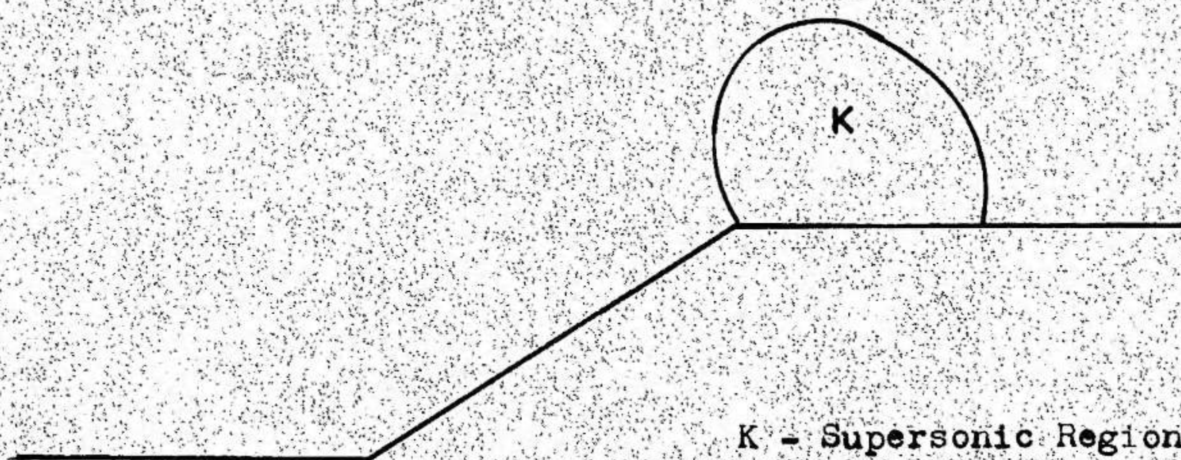
Because of the conditions in the subsonic part of the field, Ψ and its derivatives are known on a portion AP of the sonic circle. From the theory of hyperbolic differential equations this means that Ψ is determined in the region APQ where the dotted lines are the (fixed) characteristics. In particular Ψ is determined on AQ. Now if the flow at A is to turn the corner by means of a Prandtl-Meyer expansion the point A in the physical plane (Fig. (xvi)) will be represented by the characteristic AR in the hodograph plane, because the equation of AR is precisely the relation which must hold between q and θ in a Prandtl-Meyer expansion; and Ψ must be zero on AR. Ψ on AQ is already fixed and the problem is therefore overdetermined.

Nevertheless it is reasonable to suppose that the solution will be a good approximation to the flow round a wedge in the region upstream from the sonic line, in particular on the side of the wedge. For the disturbances introduced by the shoulder will not affect the oncoming flow directly. They do affect it by travelling downstream into the subsonic region and are then transmitted into the whole subsonic field, including the upstream flow. It is natural to expect that disturbances introduced in this way will be very small. In particular, the solution should give a good approximation to the pressure distribution on the wedge and to the total drag. The calculations of velocity distributions which have been carried out are in good agreement with experiment. Moreover an approximate solution might be obtained by the use of characteristics, even if this were not strictly possible analytically. Known experimental results (e.g. the location of the shock waves) could be used as a guide. In practice the gas is observed to over-expand round the corner of the wedge and has to be compressed by passing through a shock wave in order to flow along the upper surface of the wedge. Far downstream the flow would theoretically be non-uniform because of an entropy gradient across it. But the effect of viscosity would smooth out the difference in entropy and give uniform flow ultimately.

Alternative Solutions. It has now been shown that no complete, exact solution of the wedge problem is to be found by generalizing the known incompressible flow past such a profile. This was naturally selected as the one most likely to give a solution of the type sought, particularly as there is more chance of preserving straight line boundaries. It will now be shown that any exact potential solution of this type is not possible to achieve, a fact borne out experimentally by the observation of shock waves when supersonic velocities are reached.

First of all, if the flow is to remain subsonic, it cannot turn a sharp corner such as occurs at the shoulder. If the velocity at infinity upstream is low, the flow will presumably break away at the corner and be continued as a free streamline at constant pressure.

At supersonic velocities, however, the gas can turn sharp corners by means of the Prandtl-Meyer expansion. In the case of a simple expansion wave, a definite area of the physical plane is mapped on a curve in the hodograph plane and this could not be recovered from a solution for Ψ in the velocity variables. However the flow up the side of the wedge is non-uniform and the expansion would be point-localized - that is there would be no simple wave in a finite region. Therefore although the corner would be a singularity, it would be isolated and at an arbitrarily small distance from it the mapping of the hodograph and physical planes would be one-one. There is therefore no objection in principle to a flow of the type indicated in Fig.(xviii) and it might be thought that a solution could exist in the hodograph variables from which such a configuration could be recovered.



K - Supersonic Region.

Fig.(xviii)

That such a solution in the physical plane is impossible is a consequence of some interesting results derived for general flows by Nikolski and Taganov (21). Their theoretical work is now briefly summarized and applied to the present problem.

<u>The "Monotonic Law"</u>. The equations of steady, two-dimensional, potential flow are

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{uv}{a^2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad \text{(see (1.15))}$$

In terms of q, θ these become

$$(1-M^2)\left(\cos\theta\frac{\partial q}{\partial x}+\sin\theta\frac{\partial q}{\partial y}\right) - q\left(\sin\theta\frac{\partial\theta}{\partial x}-\cos\theta\frac{\partial\theta}{\partial y}\right) = 0 \qquad (4.6)$$

$$\sin\theta\frac{\partial q}{\partial x} - \cos\theta\frac{\partial q}{\partial y} + q\left(\cos\theta\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\theta}{\partial y}\right) = 0. \qquad (4.7)$$

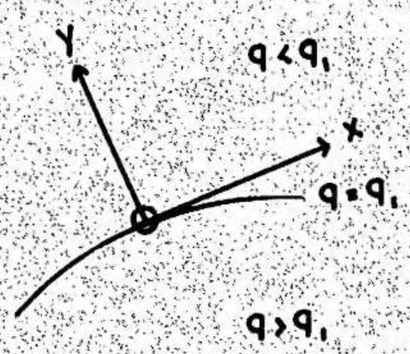


Fig.(xix)

Now consider axes chosen as in Fig. (xix) with Ox taken along the line of constant velocity and Oy directed into the region $q < q_1$. Then (4.6), (4.7) reduce to

$$(1-M_1^2)\sin\theta\,\frac{\partial q}{\partial y} - q\left(\sin\theta\,\frac{\partial\theta}{\partial x} - \cos\theta\,\frac{\partial\theta}{\partial y}\right) = 0$$

$$-\cos\theta\,\frac{\partial q}{\partial y} + q\left(\cos\theta\,\frac{\partial\theta}{\partial x} + \sin\theta\,\frac{\partial\theta}{\partial y}\right) = 0 .$$

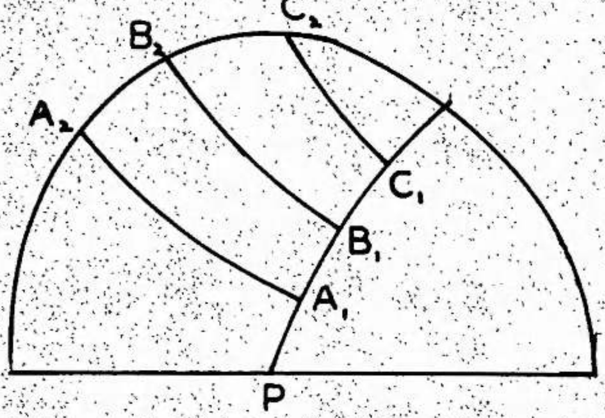
On eliminating $\frac{\partial\theta}{\partial y}$,

$$\frac{\partial\theta}{\partial x} = \frac{1-M_1^2\sin^2\theta}{q}\,\frac{\partial q}{\partial y}. \qquad (4.8)$$

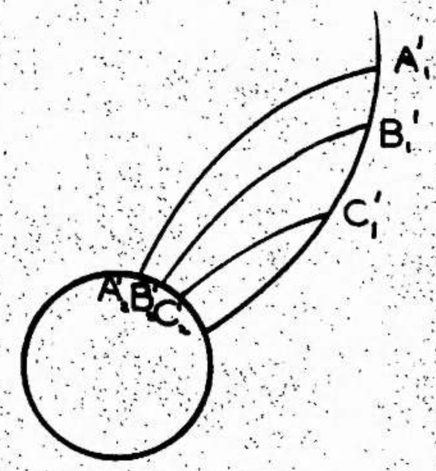
Hence for $M_1 \leq 1$, $\frac{\partial\theta}{\partial x} \leq 0$ since by choice of axes $\frac{\partial q}{\partial y} \leq 0$. In particular, for $M_1 = 1$ the following can be stated.

If a point moves along the sonic line in such a way that the region of subsonic velocity lies to the left, the velocity vector will turn monotonically in the clockwise direction.

Now in a solution of the type indicated in Fig. (xviii) let P be a point on the upper side of the wedge (Fig. (xx, a)) and let A_1, B_1, C_1 be three points on the forward facing characteristic through P. Draw the characteristics of the second family through these points to meet the sonic line in A_2, B_2, C_2. Then since θ decreases



(a)



(b)

Fig.(xx)

monotonically on the sonic line, the images A_2', B_2', C_2' in the hodograph plane will be in the order shown in Fig. (xx, b) in which all the lines shown are characteristics (except, of course, the sonic circle). Thus the images A_1', B_1', C_1' are also as shown and it is clear that $q(A_1') > q(B_1') > q(C_1')$; also that $\theta(A_1') > \theta(B_1') > \theta(C_1')$. That is, q and θ decrease monotonically along a forward facing characteristic moving away from the wedge profile. Similarly, q decreases and θ increases monotonically along a backward facing characteristic.

Now let P, Q be two points on the wedge profile and let the characteristics intersect at R as shown in Fig. (xxi). Then $\theta(R) \leq \theta(P)$ and $\theta(R) \geq \theta(Q)$

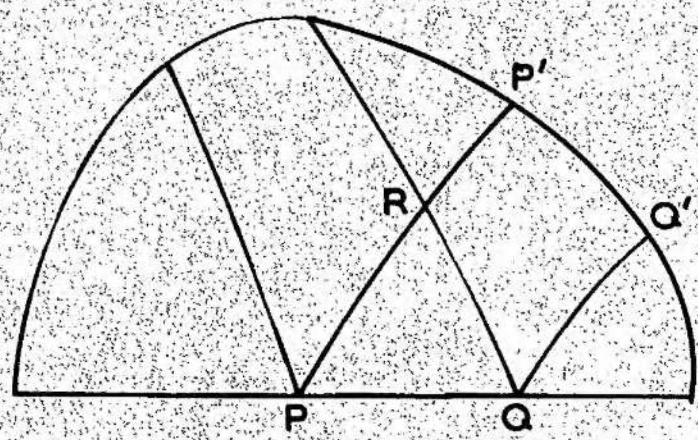


Fig.(xxi)

on applying the above result. But $\theta(P) = \theta(Q) = 0$. Hence $\theta(R) = 0$ and similarly for any point inside PQR. Now if θ is constant along a characteristic so is q; therefore the flow in the region PQR is uniform. It follows that the flow in the region P'RQQ' is a simple wave - that is, there is a relation connecting q and θ in P'R QQ'. But on P'Q', M = 1. Hence θ is constant on P'Q'. Hence u and v are constant on a definite curve and by the properties of elliptic differential equations, the flow outside the supersonic region must be uniform everywhere. Thus a contradiction has been obtained and no flow such as is indicated in Fig. (xviii) is possible.

Conclusion. A particular solution of the steady, compressible, potential flow equations in two dimensions has been obtained through a generalization of the flow of an incompressible fluid past a finite wedge. This is shown to represent the flow, always subsonic at infinity upstream, past a symmetric body. If the velocity at infinity is high enough, a supersonic region occurs and with it a limit line. The solution below any non-singular streamline $\Psi = \Psi_0$ may then be discarded, and the solution above $\Psi = \Psi_0$ will represent the flow (which may be partly supersonic) past a fixed contour whose shape is determined by $\Psi = \Psi_0$. Alternatively an approximation may be made to try and satisfy as far as possible the conditions for an expansion at the shoulder of the wedge. This would be done by using characteristics and possibly introducing shock waves. If the infinity velocity is low enough, or if the angle of the wedge is small enough, the solution contains no supersonic region but represents subsonic flow past a wedge shaped profile but with a continuous slope at the shoulder. It is noted that since the "infinity velocity" $\tau_0 = \frac{q_0^2}{q_m^2}$ is dimensionless, decreasing it can be interpreted in two ways. Either the actual velocity q_0 is lowered for a given gas or q_0 is kept constant and q_m increased, thus diminishing the compressibility of the gas.

Since the completion of the major part of this work, a number of papers have appeared in American publications describing flow past a wedge with sonic or subsonic free stream velocities. These include papers by Weinstein (27), Guderley and Yoshihara (28), Cole (29) and Trilling (30). All use the Tricomi approximation, that is a simplified form of the hodograph equations valid when M is nearly equal to one. If it is assumed that M = 1 at the shoulder and if the wedge angle is not too large, then a suitable solution of the simplified equations might be expected to represent the flow near the shoulder. Their solutions are still complicated although not so unmanageable as that obtained in the present work. However, all possess the serious disadvantage of having to make a drastic approximation for the solution near the tip. For the tip of the wedge must be a stagnation point and in quite an extensive neighbourhood of it the velocity is very far from being sonic and the Tricomi solution is inapplicable. In the approach described in the present work, the flow near the tip of the wedge can be obtained with precision; and although there is considerable complexity in conditions near the shoulder, an approximation made there, if this could be done, might give as good values for the pressure and other physical quantities near the shoulder as are predicted by the Tricomi solution.

C H A P T E R V.

NUMERICAL DETAILS.

The Difficulty of Numerical Analysis. The complicated nature of the series for Ψ_2, whether μ is rational or non-rational, makes the task of obtaining numerical data about the flow when $\tau > \tau_0$ extremely difficult. The few tables of hypergeometric functions which have been constructed are of little help in providing the numerical values of the functions required. Even for the series Ψ_1, i.e. appropriate to $\tau < \tau_0$, the range of n for which the values of $\psi_n(\tau)$ are required rapidly exceeds the limits of the tables which are available. For example, a wedge angle of 40° corresponding to $\mu = \frac{1}{9}$ requires $\psi_9(\tau)$, $\psi_{18}(\tau)$, $\psi_{27}(\tau)$ etc. For Ψ_2 no tables are applicable. The Huckel Tables (17) do give $\psi_n(\tau)$ for negative values of n of the form $-r-\frac{1}{2}$ where r is a positive integer but this is of little use in the present problem.

In order to obtain some numerical data about the flow, a section of the field near the stagnation point was evaluated. The wedge angle was taken to be 40° and the velocity at infinity to be $\tau_0 = .12$ corresponding to $M_0 = .826$ and $s_0 = -1.237858$. Apart from $\psi_9(\tau)$ and

$\psi_9'(\tau)$, which were obtained from the tables computed by Ferguson and Lighthill (25), the values of the hypergeometric functions and their derivatives had to be calculated individually. Some of the results were used subsequently to determine some details about the corresponding flow with a wedge angle of 20°.

<u>Scheme of Work</u>. The fundamental equation in obtaining the flow in the physical plane is (1.28). The functions required present no difficulties apart from $\psi_n(\tau)$ and $\psi_n'(\tau)$ for $n = 9, 18, 27, \ldots\ldots$ It is important to observe the speed of convergence of the series. This is obtained by examination of the asymptotic forms of the hypergeometric functions. For large n, $\psi_n(\tau)e^{-ns_o} \sim e^{n(s-s_o)}$ so that the rate of convergence decreases as τ increases. The values of $\psi_n(\tau)$ and $\psi_n'(\tau)$ were found for $\tau = .04, .06, .08$ and $.10$ and for $n = 18, 27, \ldots\ldots$ up to 63 in the case $\tau = .10$. They were calculated by means of asymptotic formulae involving Bessel functions - the values of $\psi_9(\tau)$ and $\psi_9'(\tau)$ were already available. It was impossible to proceed further than this stage because the Bessel functions were so small that the hypergeometric functions could not be calculated with any accuracy. But the terms in the series arising from them remained significant

because of the powerful weighting term e^{-ns_o} for $n = 9, 18 \ldots$. It was observed, however, that the hypergeometric functions were very nearly in a geometric progression and the remainder terms in the series for z were summed to infinity as a complex geometric series.

The calculation of the hypergeometric functions was carried out by means of the asymptotic formulae given by Cherry (23). These formulae are

$$\chi_\nu(\gamma) \sim N\left(\frac{\nu}{h_\nu}\right)^{\frac{1}{2}}\left[J_\nu(\nu t)\left(1+\frac{q_2}{\nu^2}+\frac{q_4}{\nu^4}+\cdots\right)+t\,J_\nu'(\nu t)\left(\frac{q_1}{\nu}+\frac{q_3}{\nu^3}+\cdots\right)\right] \quad (5.1)$$

$$\frac{2\gamma}{\nu}\chi_\nu'(\gamma) \sim N\left(\frac{\nu}{h_\nu}\right)^{\frac{1}{2}}\left[J_\nu(\nu t)\left(\frac{q_1^*}{\nu}+\frac{q_3^*}{\nu^3}+\cdots\right)+t\,J_\nu'(\nu t)\left(q_0^*+\frac{q_2^*}{\nu^2}+\frac{q_4^*}{\nu^4}+\cdots\right)\right] \quad (5.2)$$

where $N = (1-\gamma)^{\frac{1}{4}-\frac{1}{2}\beta}\left(\frac{1-t^2}{1-\gamma/\gamma_s}\right)^{\frac{1}{4}}$. $t, q_1, q_2, \ldots q_1^*, q_2^*, \ldots$ are all functions of γ which are tabulated in Cherry's paper. h_ν is a function of ν given by the asymptotic formula

$$\log\,(2\pi h_\nu)^{\frac{1}{2}} \sim 1.14344\,36154\,\nu + \frac{0.24805825}{\nu} - \frac{0.044461}{\nu^3} - \frac{0.0346}{\nu^5} + \cdots$$

$\psi_\nu(\gamma)$, $\psi_\nu'(\gamma)$ are obtained from (5.1), (5.2) by means of the relations

$$(\nu-1)\ (1-\gamma)^{-\beta}\psi_\nu(\gamma) = \nu\,\chi_\nu(\gamma) - \frac{2\gamma\,\chi_\nu'(\gamma)}{\nu}$$

$$(\nu+1)\ (1-\gamma)^{\beta}\chi_\nu(\gamma) = \nu\,\psi_\nu(\gamma) + \frac{2\gamma\,\psi_\nu'(\gamma)}{\nu}.$$

The required Bessel functions $J_\nu(x)$ are tabulated in (24). x is given in intervals of .01. To obtain the

maximum accuracy νt was obtained to four places of decimals and the corresponding value of $J_\nu(\nu t)$ was found by interpolation. $J_\nu'(x)$ was found by numerical differentiation of the tables for values of x in the neighbourhood of νt and then $J_\nu'(\nu t)$ was found by interpolating between these values. In this way the functions $\psi_n(\tau) \pm \frac{2\tau}{n}\psi_n'(\tau)$, the combinations required in (1.28), were obtained.
Values of $A_n(\tau)$, $B_n(\tau)$ are given in Tables 1, 2 where

$$A_n(\tau) = \left\{ \psi_n(\tau) + \frac{2\tau}{n}\psi_n'(\tau)\right\} e^{-ns_0} \qquad (5.3)$$

$$B_n(\tau) = \left\{ \psi_n(\tau) - \frac{2\tau}{n}\psi_n'(\tau)\right\} e^{-ns_0} \qquad (5.4)$$

The calculation of the functions $\frac{n e^{(n+1)i\theta}}{n+1}$ and $\frac{n e^{(n+1)i\theta}}{n-1}$ is straightforward and was done for $\theta = 0^\circ$, 5°, 10°, 15° and 20°. A table of values of $E(\tau)$ with $\frac{c^2}{q_m} = 1$ was also obtained and hence z was evaluated by means of substitution in (3.20). n now replaces $\frac{n}{\mu}$ where it occurs in (3.20), that is, the n in (5.3), (5.4) etc. has to take the values 9, 18, ... corresponding to $\mu = 1/9$.

<u>Correction Terms</u>. When $\tau = .04$ and $.06$, five (or fewer) terms were found to give z to the required accuracy. However, for $\tau = .08$ and $.10$, the last term which could be

evaluated from the Bessel Function Tables was found to be still significant due to the slowness of the convergence. The expressions $\frac{A_{n+q}(\tau)}{A_n(\tau)}$ and $\frac{B_{n+q}(\tau)}{B_n(\tau)}$ were evaluated and found to be practically constant. A remainder term was obtained by regarding the last term which could be calculated as the first term of a geometric progression with a complex common ratio and the sum to infinity was evaluated and added to the other terms.

<u>Discussion of the Results.</u> The values of x and y are set out in Table 3 and plotted in Fig. (xxii). It is observed that the lines of constant τ are smooth curves in general qualitative agreement with those obtained experimentally and described by Pack (16). They are (as they must be) perpendicular to the straight streamlines. Some auxiliary results give the velocity distribution on the axis and up the wedge side for a 20° wedge (Table 4). It is interesting to compare the ratios $\frac{x(\tau)}{r(\tau)}$ for a 20° and 40° wedge where x denotes the distance from the origin along the axis and $r(\tau)$ the distance up the wedge side. It is noted from Figs. 7 and 8 in (16) that experimentally this ratio is observed to be appreciably greater for 40° than for 20° wedges and this result is apparent from the calculated values (Table 5).

It is realized that in themselves these results are not of great physical interest. They were calculated in order to confirm that the solution did give the general pattern of flow past a wedge. Although the streamline $\Psi = 0$ can be determined from the solution directly, conditions in the interior of the flow cannot be obtained by inspection of the analytical solution. The good agreement between Fig. (xxii) and Fig. 8 in (16) establishes the nature of the general flow pattern and gives some concrete justification for the assertion made in Chapter IV that the flow up the wedge obtained theoretically is a good estimate of that which occurs in practice. Any more elaborate computation of the field downstream from the shoulder can therefore be undertaken with some confidence. The difficulties encountered in the present work would seem, however, to prevent further calculations, at least until tables of Bessel functions for higher values of ν and giving more places are available. It might be that a more satisfactory approach would be to evaluate numerically the integral form of the stream function obtained in Chapter III. The initial difficulties would be considerable but the problem of the slow convergence near $\Upsilon = \Upsilon_0$ would be avoided and the whole subsonic field could be calculated apart from the neighbourhood of the axis on which $\theta = 0$.

<u>The Second Problem</u>. The other numerical problem which was evaluated was the flow corresponding to

$$\Psi = -\psi_{-13.5}(\tau)\sin 13.5\theta + 10^6\psi_{-4.5}(\tau)\sin 4.5\theta$$

A general description of this flow has been given in Chapter IV. The numerical details are straightforward. The Huckel Tables (17) give values of $Y_k(\tau)$ for the required values of k , i.e. k = -4.5 and -13.5 where $Y_k(\tau) = \tau^{-\frac{k}{2}}\psi_k(\tau) = F(a_k, b_k; k+1; \tau)$. There is an error in the tables of the derivatives of $Y_k(\tau)$. The function tabulated as $\frac{dY_k(\tau)}{d\tau}$ is actually $\left(\frac{dY_k(\tau)}{d\tau}\right)\Big/\left(\pm\frac{5k}{4}\right)$, the plus or minus sign being taken according as k is negative or positive.* The streamline $\Psi = 0$ was plotted for non-zero values of θ by means of (4.5) to obtain the values of τ, θ and then (1.28) was used to obtain the corresponding values of x, y. These are set out in Table 6. In addition the values of Ψ, x and y for various values of τ and θ were calculated. These results are contained in Table 7. The values obtained include a number of points for which $\Psi < 0$. The purpose of this was to provide a check on the working - in all cases for which $\Psi < 0$, the corresponding point (x, y) was found to lie below $\Psi = 0$ -

* I am grateful to Prof. T. M. Cherry for informing me of this error in the tables.

and also because of the general interest of the secondary flow below $\Psi = 0$.

A general picture of the flow in the physical and hodograph planes was given in Figs. (xiv) and (xv) for convenience of reference. The accurate plotting of the physical plane is now given in Fig. (xxiii). Due to the wide spacing between the points (x,y) corresponding to the values of τ, θ taken it is only possible to plot the points corresponding to τ = .04, .05 and .06 on one figure. The streamline $\Psi = 0$ in the hodograph plane is plotted in Fig. (xxiv). In this figure the θ scale is multiplied by a factor of twelve.

No attempt was made to obtain more than a few significant figures in these calculations. The difficulty of obtaining any higher accuracy is due to the small number of places given in the Huckel Tables and aggravated by the necessity of applying a correction term. Maximum accuracy within the limits of the tables was, however, preserved at all stages, and the results obtained are certainly sufficient to guarantee the general pattern of the flow.

TABLE I.

Values of $A_n(\tau)$

$n \backslash \tau$	.04	.06	.08	.10
9	10^{-2}x4.18978	10^{-1}x1.96593	10^{-1}x5.353835	10^{-1}x10.731491
18	10^{-4}x9.2214	10^{-2}x2.0794	10^{-1}x1.5773	10^{-1}x6.4716
27	10^{-5}x2.030	10^{-3}x2.1991	10^{-2}x4.6458	10^{-1}x3.8959
36		10^{-4}x2.325	10^{-2}x1.3683	10^{-1}x2.3459
45			10^{-3}x4.0294	10^{-1}x1.4123
54			10^{-3}x1.188	10^{-1}x.8502
63				10^{-1}x.5122

TABLE 2.

Values of $B_n(\tau)$

$n \backslash \tau$	.04	.06	.08	.10
9	$10^{-3}\times 2.378$	$10^{-1}\times .17972$	$10^{-1}\times .7051234$	$10^{-1}\times 1.923654$
18	$10^{-4}\times .5306$	$10^{-2}\times .1940$	$10^{-1}\times .2142$	$10^{-1}\times 1.2152$
27	$10^{-5}\times 1.18$	$10^{-3}\times .2072$	$10^{-2}\times .6390$	$10^{-1}\times .7436$
36		$10^{-4}\times .220$	$10^{-2}\times .1893$	$10^{-1}\times .4524$
45			$10^{-2}\times .5614$	$10^{-1}\times .2742$
54			$10^{-3}\times .165$	$10^{-1}\times .1659$
63				$10^{-1}\times .1002$

T A B L E 3.

Values of (x, y) for various values of (γ, θ)

The value in the row corresponding to 20° is r, the distance up the wedge side.

$\theta \backslash \gamma$	.04	.06	.08	.10
0°	-.127, 0	-.542, 0	-1.622, 0	-4.996, 0
5°	-.096, .091	-.380, .429	-.861, 1.436	-.898, 3.798
10°	-.021, .135	-.050, .559	.058, 1.372	.492, 2.239
15°	.061, .113	.241, .424	.579, .876	1.001, 1.258
20°	.122	.445	.925	1.385

TABLE 4.

Values of $x(\tau)$ and $r(\tau)$ for 20° wedge

τ	$x(\tau)$	$r(\tau)$
.04	.00258	.00256
.06	.0486	.0476
.08	.349	.295
.10	1.806	.873

TABLE 5.

Values of $\frac{x(\tau)}{r(\tau)}$ for 20° and 40° wedges

τ	20°	40°
.04	1.01	1.04
.06	1.02	1.22
.08	1.18	1.75
.10	2.07	3.61

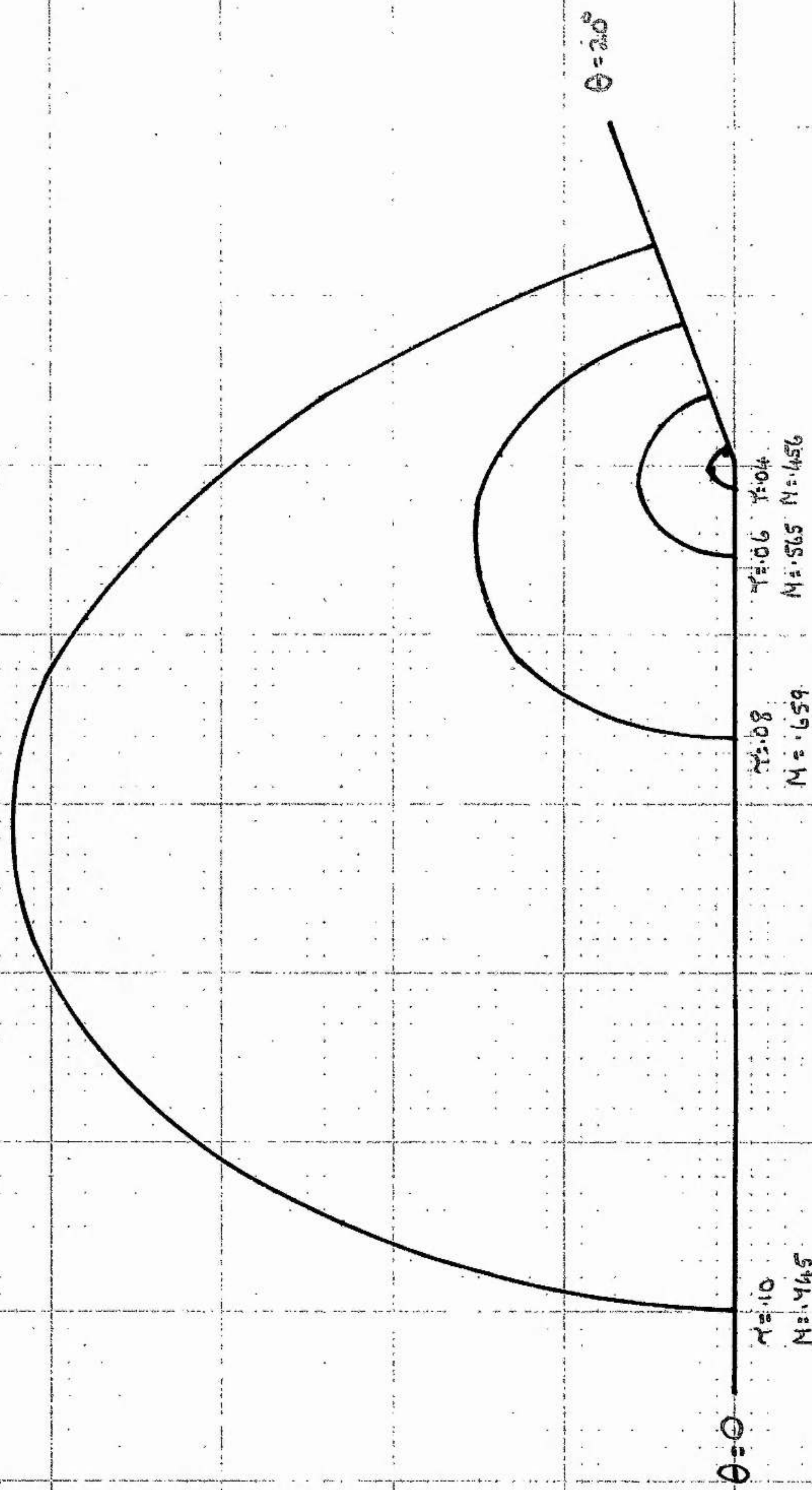


Fig. (xxii)

TABLE 6.

Points on the curved part of $\Psi = 0$.

The values of x and y have been multiplied by 10^{-8}.

θ is given in degrees.

Υ	θ	x	y
.04	-12.20	278.5	-54.9
.05	-10.71	73.76	-10.13
.06	- 8.20	25.88	- 1.85
.07	- 3.32	10.27	- .06

T A B L E 7.

Position coordinates for values of r, θ in model solution.
(x, y) is given first and below it the corresponding value of Ψ.
Values of x, y have been multiplied by 10^{-8}.

r/θ	0°	-2°	-4°	-6°	-8°	-10°	-12°	-13°
.04	-185, 0 0	-154, 124 .23	-69.0, 215 .40	47.1, 245 .47	164, 206 .43	249, 104 .27	279, -37.3 .027	275, -113 -.11
.05	-21.8, 0 0	-14.8, 25.3 .050	4.08,42.4 .086	29.6, 45.2 .096	54.4, 31.3 .074	71.0, 2.60 .023	73.8, -35.2 -.048	69.2, -54.9 -.088
.06	4.75, 0 0	6.73, 5.35 .011	12.1, 8.10 .018	19.1, 6.24 .015	25.4, -.88 .0019	28.8, -12.1 -.021	27.3, -26.9 -.051	24.5, -29.4 -.072
.07	8.61, 0 0	9.25, .27 .0006	10.91, -.49 -.0008	13.0, -3.00 -.0060	14.5, -7.51 -.015	14.6, -13.7 -.029	12.6, -20.7 -.044	10.8, -24.1 -.053
.08	8.11, 0 0	8.30, -1.16 -.0026	8.78, -2.35 -.0063	9.25, -5.16 -.012	9.31, -8.40 -.019	8.57, -12.3 -.028	6.77, -16.4 -.038	5.45, -18.3 -.043
.09	6.36, 0 0	6.46, -1.27 -.0036	6.70, -2.83 .0078	6.88, -4.86 -.013	6.73, -7.42 -.019	6.02, -10.4 -.026	4.58, -13.4 -.033	3.56, -14.8 -.036

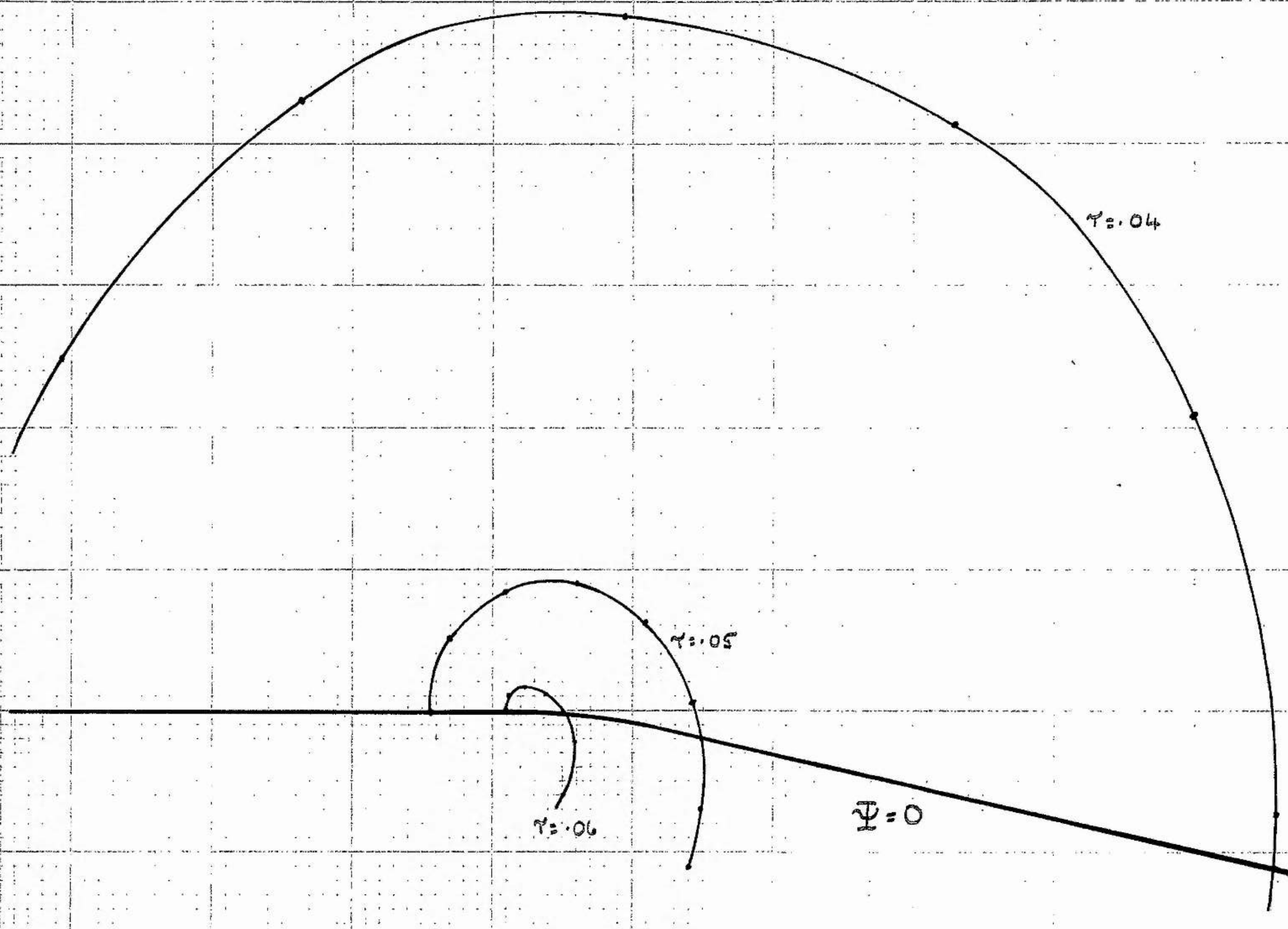
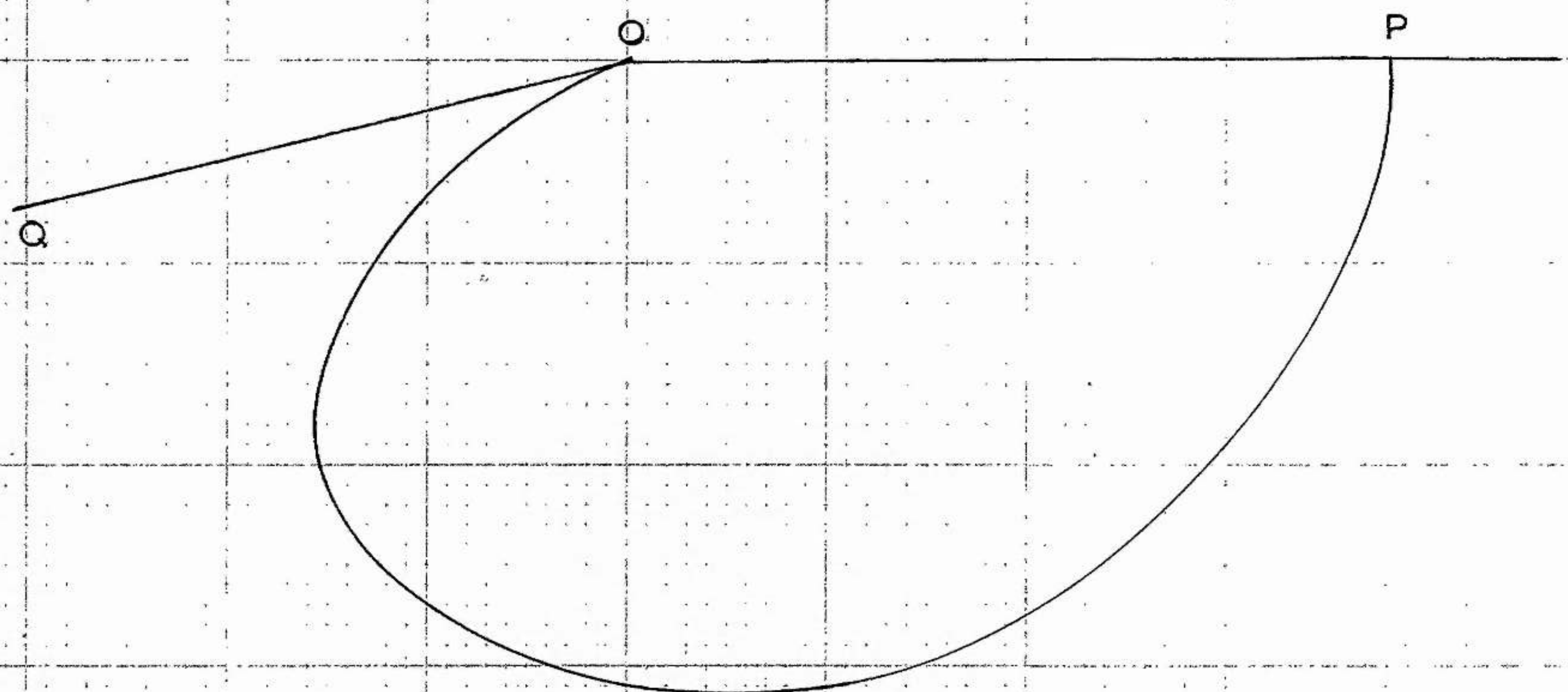


Fig. (xxiii)



OP is $\theta = 0$

OQ represents $\theta = -\frac{\pi}{13.5}$ on the magnified θ scale.

Fig. (xxiv)

B I B L I O G R A P H Y.

(1) MACKIE, A.G. & PACK, D.C. Proc. Camb. Phil Soc. 48, (1952), 178.

(2) LIEPMANN, H.W. & PUCKETT, A.E. Introduction to Aerodynamics of a Compressible Fluid. Wiley (1947).

(3) MOLENBROEK, P. Arch. Math. Phys., Lpz. (2), 9 (1890), 157.

(4) CHAPLYGIN, A. Ann. Sci. Univ. Moscow, 21, (1904), 1. (Ministry of Supply, R.T.P. translation No. 1267).

(5) CRAGGS, J.W. Proc. Camb. Phil. Soc. 44 (1948), 360.

(6) TAYLOR, G.I. Some Cases of Flow of Compressible Fluids. R. and M. no. 1382 (1930).

(7) von RINGLEB, F. Z. Angew. Math. Mech. 20 (1940). (Ministry of Supply R.T.P. translation No. 1609).

(8) TEMPLE, G. & YARWOOD, J. Compressible Flow in a Convergent-Divergent Nozzle. R. & M. No. 2077 (1942).

(9) von KARMAN, T. Jour. Aero. Sci. 8 (1941), 337.

(10) WILLIAMS, J. Quart. Jour. Math. 20 (1949), 129.

(11) GOLDSTEIN, S., LIGHTHILL, M.J. & CRAGGS, J.W. Quart. Jour. Mech. Appl. Math. 1 (1948), 344.

(12) LIGHTHILL, M.J. Proc. Roy. Soc. A, 191, (1947), 323.

(13) LIGHTHILL, M.J. Proc. Roy. Soc. A, 191, (1947), 341.

(14) LIGHTHILL, M.J. Proc. Roy. Soc. A, 191, (1947), 352.

(15) CHERRY, T. M. Proc. Roy. Soc. A, 192, (1947), 45.

(16) PACK, D.C. Rep. Memor. Aero. Res. Coun., Lond. No. 2321 (1949).

(17) HUCKEL, V. Tech. Notes Nat. Adv. Comm. Aero., Wash., no. 1716 (1948).

(18) TOLLMIEN, W. Z. Angew. Math. Mech. 21 (1941).

(19) BUSEMANN, A. Tech. Notes Nat. Adv. Comm. Aero., Wash., no. 1858 (1949).

(20) MACCOLL, J.W. Sixth Int. Congr. Appl. Mech. (Paris, 1946).

(21) NIKOLSKI, A.A. & TAGANOV, G.I. Tech. Memo. Nat. Adv. Comm. Aero., Wash., no. 1213 (1949).

(22) COURANT, R & FRIEDRICHS, K.O. Supersonic Flow and Shock Waves. Interscience, New York(1948).

(23) CHERRY, T.M. Proc. Roy. Soc. A, 202 (1950), 507.

(24) HARVARD COMPUTATION LABORATORY. Tables of Bessel Functions of the First Kind. (Harvard University Press, 1949).

(25) FERGUSON, D.F. & LIGHTHILL, M.J. Proc. Roy. Soc. A, 192 (1947), 135.

(26) GRIFFITH, W.C. Transonic Flow. Princeton University, Dept. of Physics, Tech. Report II-7 (1950).

(27) WEINSTEIN, A. Transonic Flow and Generalized Axially Symmetric Potential Theory. Proceedings of the Naval Ordnance Laboratory's Aeroballistic Research Symposia (1950).

(28) GUDERLEY, G. & YOSHIHARA, H. Jour. Aero. Sci. 17 (1950), 723.

(29) COLE, J.D. Jour. Math. Phys. 30 (1951), 79.

(30) TRILLING, L. Transonic Flow Past a Wedge at Zero Angle of Attack. Dept. of Aero. Eng., Massachusetts Institute of Technology (1952).